Last name ______________________

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LARSON—MATH 656—Homework #7 (h07)
Gallai-Edmonds-Decomposition.

Notes

1. A vertex v in a graph is either (1) covered by every maximum matching (set B), or (2) not covered by every maximum matching (set D). A vertex in B either (1) has a neighbor outside B (set A) or (2) does not (set C). The **Gallai-Edmonds Decomposition** is the partition of $V(G)$ into sets C, A and D.

2. One (efficient) algorithm for finding the Gallai-Edmonds Decomposition is simply to test each vertex v to see whether it is in D (so test if $\alpha'(G-v)=\alpha'(G)$). Then A must be the vertices adjacent to the vertices in D and C must be the remaining vertices ($C=V-A-D$).

3. (**Gallai-Edmonds Structure Theorem**). Let A, C, D, be the sets in the Gallai-Edmonds Decomposition of a graph G. Let $G_1,\ldots,G_k$ be the components of $G[D]$. If M is a maximum matching in G then:

   (a) M covers C and matches A into distinct components of $G[D]$.

   (b) Each G_i is factor-critical and M restricts to a near-perfect matching on G_i,

   (c) If $S \subseteq A$ is non-empty then $N_G(S)$ has a vertex in at least $|S|+1$ of $G_1,\ldots,G_k$,

   (d) $def(A)=def(G)=k-|A|$.

4. The **structure of West's proof**, given a maximum matching M of a graph G with decomposition sets, C, A, D, is:

   (a) Define T as in the proof of the Berge-Tutte formula proof (we'll also need facts about the auxiliary graph $H(T)$),

   (b) We also know:

      i. All components of $G-T$ are factor-critical (and hence odd),

      ii. Any maximum matching matches T to one vertex in each of $|T|$ components of $G-T$ (in particular M),

   (c) Define $R \subseteq T$ to be a maximal subset with $|N_{H(T)}(R)|=|R|$,

   (d) Let R' be the union of the components corresponding to the vertices R matches in $H(T)$ with respect to M,

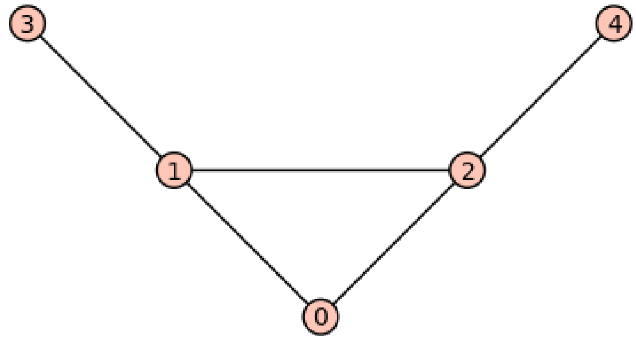
   (e) Argue that $R \cup R' \subseteq C$ (and later $R \cup R' = C$),

   (f) Let $D' = V(G)-T-R'$ and argue $D=D'$, and

   (g) Argue $A=T-R$.

Problems

1. Prove: Given a graph G and a vertex v, there is a maximum matching in G which does not cover v if and only if $\alpha'(G-v)=\alpha'(G)$.



2. Gallai-Edmonds Decomposition and Proof Notes for the **Bull** graph G.

   (a) Find the Gallai-Edmonds Decomposition (that is, find sets C, A, D).

   (b) Find a maximum matching M (not necessarily unique).

   (c) Check that M covers C and matches A into distinct components of $G[D]$.

   (d) Check that each G_i is factor-critical and M restricts to a near-perfect matching on G_i

   (e) Find the set T (not necessarily unique) as in the proof of the Berge-Tutte formula.

   (f) Check that all components of $G - T$ are factor-critical (and hence odd).

   (g) Check that M matches T to one vertex in each of $|T|$ components of $G - T$,

   (h) Find the auxiliary graph $H(T)$.

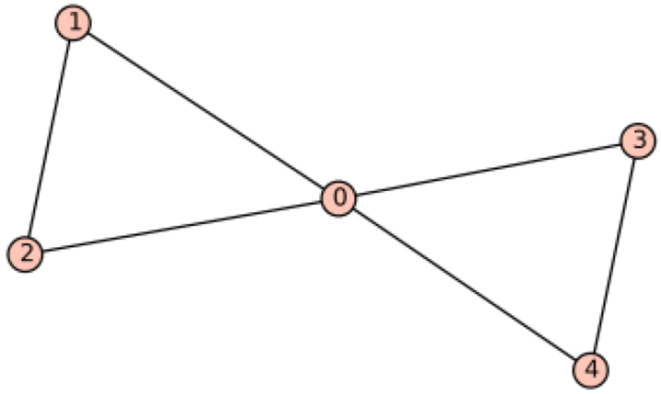
   (i) Find $R \subseteq T$ with $|N_{H(T)}(R)| = |R|$ and maximal with respect to this condition.

   (j) Find R', the union of the components corresponding to the vertices R matches in $H(T)$ with respect to M.

   (k) Check that $R \cup R' = C$.

   (l) Find $D' = V(G) - T - R'$, and check that $D = D'$.

   (m) Check that $A = T - R$.



3. Gallai-Edmonds Decomposition and Proof Notes for the **Bow Tie** graph G.

   (a) Find the Gallai-Edmonds Decomposition (that is, find sets C, A, D).

   (b) Find a maximum matching M (not necessarily unique).

   (c) Check that M covers C and matches A into distinct components of $G[D]$.

   (d) Check that each G_i is factor-critical and M restricts to a near-perfect matching on G_i

   (e) Find the set T (not necessarily unique) as in the proof of the Berge-Tutte formula.

   (f) Check that all components of $G - T$ are factor-critical (and hence odd).

   (g) Check that M matches T to one vertex in each of $|T|$ components of $G - T$,

   (h) Find the auxiliary graph $H(T)$.

   (i) Find $R \subseteq T$ with $|N_{H(T)}(R)| = |R|$ and maximal with respect to this condition.

   (j) Find R', the union of the components corresponding to the vertices R matches in $H(T)$ with respect to M.

   (k) Check that $R \cup R' = C$.

   (l) Find $D' = V(G) - T - R'$, and check that $D = D'$.

   (m) Check that $A = T - R$.