Last name ______________________

First name ______________________

LARSON—MATH 656—CLASSROOM WORKSHEET 20
Gallai-Edmonds-Decomposition.

Organizational Notes

1. Don't forget to send your Notes / Classroom worksheet after each class (make the email subject useful: like "Math 656 c20 notes").
2. The VCU Discrete Math Seminar is every Wednesday.
3. $h06$ is due today (#3.3.1, 3.3.2, 3.3.3, 3.3.6, 3.3.10).
4. Read ahead! Next up we'll talk about Petersen's Theorem (Corollary to Tutte's Theorem in Sec 3.3) and then Network Flow problems (Sec. 4.3)

Concepts & Notation

- factor-critical graph, near-perfect matching, Edmonds-Gallai Decomposition (West paper).
- Petersen's Theorem (Sec. 3.3).
- Network Flows (Sec. 4.3).

Review

1. (**Theorem**) (Berge-Tutte Formula) $\nu = \frac{1}{2}(n - def(G))$.
2. (**Theorem**) (Tutte's Theorem) A graph G has a perfect matching if and only if for every $S \subseteq V(G)$ $o(G - S) \leq |S|$.
3. What is a *factor-critical graph*?
4. What is a *near-perfect* matching?

Notes

1. A vertex v in a graph is either (1) covered by every maximum matching (set B), or (2) not covered by every maximum matching (set D). A vertex in B either (1) has a neighbor outside B (set A) or (2) does not (set C). The **Gallai-Edmonds Decomposition** is the partition of $V(G)$ into sets C, A and D.
2. One (efficient) algorithm for finding the Gallai-Edmonds Decomposition is simple to test each vertex v to see whether it is in D. Then A must be the vertices adjacent to the vertices in D and C must be the remaining vertices ($C = V - A - D$).
3. Find the Gallai-Edmonds Decomposition for a graph with a perfect matching.
4. Find the Gallai-Edmonds Decomposition for P_3.

5. Find the Gallai-Edmonds Decomposition for S_4.

6. Find the Gallai-Edmonds Decomposition for the house graph.

7. Find the Gallai-Edmonds Decomposition for the graph formed by the join of $3K_3$ and P_2.

8. (**Gallai-Edmonds Structure Theorem**). Let A, C, D, be the sets in the Gallai-Edmonds Decomposition of a graph G. Let $G_1, \ldots, G_k$ be the components of $G[D]$. If M is a maximum matching in G then:

   (a) M covers C and matches A into distinct components of $G[D]$.

   (b) Each G_i is factor-critical and M restricts to a near-perfect matching on G_i,

   (c) If $S \subseteq A$ is non-empty then $N_G(S)$ has a vertex in at least $|S| + 1$ of $G_1, \ldots, G_k$,

   (d) $def(A) = def(G) = k - |A|$.

   The structure of West's proof, given a maximum matching M of a graph G with decomposition sets, C, A, D, is:

   (a) Define T as in the proof of the Berge-Tutte formula proof (we'll also need facts about the auxiliary graph $H(T)$),

   (b) We also know:

      i. All components of $G - T$ are factor-critical (and hence odd),

      ii. Any maximum matching matches T to one vertex in each of T components of $G - T$ (in particular M),

   (c) Defire $R \subseteq T$ to be a maximl subset with $N_{H(T)}(R)| = |R|$,

   (d) Let R' be the union of the components corresponding to the vertices R matches in $H(T)$ with resoect to M,

   (e) Argue that $R \cup R' \subseteq C$ (and later $R \cup R' = C$),

   (f) Let $D' = V(G) - T - R'$ and argue $D = D'$, and

   (g) Argue $A = T - R$.

9. What does the Gallai-Edmonds Structure Theorem say for a graph with a perfect matching? Find a maximum matching M, a maximal maximum deficiency set T and check the theorem claims.

10. What does the Gallai-Edmonds Structure Theorem say for P_3? Find a maximum matching M, a maximal maximum deficiency set T and check the theorem claims.

11. What does the Gallai-Edmonds Structure Theorem say for S_4? Find a maximum matching M , a maximal maximum deficiency set T and check the theorem claims.

12. What does the Gallai-Edmonds Structure Theorem say for the house graph? Find a maximum matching M, a maximal maximum deficiency set T and check the theorem claims.

13. What does the Gallai-Edmonds Structure Theorem say for the graph formed by the join of $3K_3$ and P_2? Find a maximum matching M, a maximal maximum deficiency set T and check the theorem claims.