DISJOINT TYPE GRAPHS WITH NO SHORT ODD CYCLES

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Abstract. In this note, we provide a proof of a technical result of Erdős and Hajnal about the existence of disjoint type graphs with no odd cycles. We also prove that this result is sharp in a certain sense.

The purpose of this note is to provide a proof of a result of Erdős and Hajnal about the existence of disjoint type graphs with no short odd cycles. As far as we know, a proof of this result has never been published, though forms of it are stated in a number of publications (cf. [2, Theorem 7.4] and [3, Lemma 1.1(d)]). If \( \kappa \) is an uncountable cardinal, then graphs of this form provide, again as far as we know, the only known ZFC examples of graphs with size and chromatic number \( \kappa \) and arbitrarily high odd girth.

Before we state and prove the main result, we need some definitions and conventions. First, if \( n \) is a positive integer, we will sometimes think of elements of \([\text{Ord}]^n\) as strictly increasing sequences of length \( n \). So, for instance, if \( a, b \in [\text{Ord}]^n \) and \( i < n \), then \( a(i) \) is the unique element \( \alpha \in a \) such that \( |a \cap \alpha| = i \). All graphs considered here will be simple undirected graphs. If \( G \) is a graph, then \( V(G) \) denotes its vertex set and \( E(G) \) denotes its edge set.

Definition 1. Let \( n \) be a positive integer. A disjoint type of width \( n \) is a function \( t : 2^n \to 2 \) such that

\[ |t^{-1}(0)| = |t^{-1}(1)| = n. \]

If \( a, b \in [\text{Ord}]^n \) are disjoint and \( a \cup b \) is enumerated in increasing order as \( \{ \alpha_i \mid i < 2n \} \), then we say that the type of \( a \) and \( b \) is \( t \), denoted \( \text{tp}(a, b) = t \), if

\[ a = \{ \alpha_i \mid i \in t^{-1}(0) \} \]

and

\[ b = \{ \alpha_i \mid i \in t^{-1}(1) \}. \]

Let \( \hat{t} \) denote the disjoint type of width \( n \) denoted by letting \( \hat{t}(i) = 1 - t(i) \) for all \( i < 2n \). It is evident that, if \( a, b \in [\text{Ord}]^n \) are disjoint and \( \text{tp}(a, b) = t \), then \( \text{tp}(b, a) = \hat{t} \).

A type \( t \) of width \( n \) will sometimes be represented by a binary string of length \( 2n \) in the obvious way. We will particularly be interested in the following family of types.

Definition 2. Let \( 1 \leq s < n < \omega \). Then \( t^n_s \) is the disjoint type of width \( n \) whose binary sequence representation consists of \( s \) copies of ‘0’, followed by \( n - s \) copies of ‘01’, followed by \( s \) copies of ‘1’. More formally, \( t^n_s \) is defined by letting, for all
\[ i < 2n, \]
\[
\begin{align*}
    t_s^n(i) &= \begin{cases} 
        0 & \text{if } i < s \\
        0 & \text{if } s \leq i < 2n - s \text{ and } i - s \text{ is even} \\
        1 & \text{if } s \leq i < 2n - s \text{ and } i - s \text{ is odd} \\
        1 & \text{if } i \geq 2n - s. 
    \end{cases}
\end{align*}
\]

For example, \( t_2^5 = 0001010111. \)

**Definition 3.** Suppose that \( n \) is a positive integer, \( \beta \) is an ordinal, and \( t \) is a disjoint type of width \( n \). The graph \( G(\beta, t) \) is defined as follows. Its vertex set is \( V(G(\beta, t)) = [\beta]^n \). Given \( a, b \in [\beta]^n \), we put the edge \( \{a, b\} \) into \( E(G(\beta, t)) \) if and only if \( a \) and \( b \) are disjoint and \( tp(a, b) \in \{t, \hat{t}\} \).

Before we get to our main result, we need a basic lemma. Given a function \( f \) from a natural number to \( \mathbb{Z} \), let \( \max(f) \) and \( \min(f) \) denote the maximum and minimum values attained by \( f \), respectively.

**Lemma 4.** Suppose that \( k \) is a positive integer and \( f : k \to \mathbb{Z} \) is a function such that
\[
\begin{itemize}
    \item \( f(0) = 0 \) and
    \item \( |f(i + 1) - f(i)| = 1 \) for all \( i < k \).
\end{itemize}
Then \( \max(f) - \min(f) < k \).

**Proof.** The proof is by induction on \( k \). If \( k = 1 \), then \( \max(f) = \min(f) = f(0) = 0 \). Suppose that \( k > 0 \) and we have proven the lemma for \( k - 1 \). Fix \( f : k \to \mathbb{Z} \), and let \( f^- = f \upharpoonright (k - 1) \). If \( f(k) - f(k - 1) = 1 \), then we have \( \max(f^-) \leq \max(f^-) + 1 \) and \( \min(f) = \min(f^-) \), so, applying the induction hypothesis to \( f^- \), we obtain
\[
\max(f) - \min(f) \leq 1 + (\max(f^-) - \min(f^-)) < 1 + (k - 1) = k.
\]
If \( f(k) - f(k - 1) = -1 \), then we have \( \max(f) = \max(f^-) \) and \( \min(f) \geq \min(f^-) - 1 \), so
\[
\max(f) - \min(f) \leq (\max(f) - \min(f^-)) + 1 < (k - 1) + 1 = k.
\]

We are now ready for the main result of this note. The proof is rather technical; we recommend that the reader first draw some pictures to convince themselves of the truth of the theorem in the special case \( s = 1 \), \( n = 3 \) (this pair does not satisfy \( n > 2s^2 + 3s + 1 \), but the conclusion of the theorem still holds). This will help the reader to get a feel for the problem and motivate the calculations in the proof. We also note that the lower bound of \( 2s^2 + 3s + 1 \) is probably not optimal and can likely be improved with a more careful analysis. Since a precise lower bound for \( n \) is not necessary for our desired applications (cf. [5]), the primary interest of the result for us is the fact that such a lower bound exists at all.

**Theorem 5.** Suppose that \( s \) and \( n \) are positive integers with \( n > 2s^2 + 3s + 1 \), and suppose that \( \beta \) is an ordinal. Then the graph \( G(\beta, t_s^n) \) has no odd cycles of length \( 2s + 1 \) or shorter.

**Proof.** Let \( t = t_s^n \), \( V = [\beta]^n \), \( E = E(G(\beta, t)) \), and \( G = G(\beta, t) = (V, E) \). We begin by making some preliminary observations. If \( \{a, b\} \in E \), then either \( tp(a, b) = t \) or \( tp(a, b) = \hat{t} \). If \( tp(a, b) = t \), then, for all \( i \) with \( s < i < n \), we have
\[
b(i - s - 1) < a(i) < b(i - s).\]
If \( \text{tp}(a,b) = \hat{t} \), then, for all \( i < n - s - 1 \), we have
\[
b(i + s) < a(i) < b(i + s + 1).
\]
Suppose that \( k \) is a positive integer and \( P = \langle a_0, \ldots, a_k \rangle \) is a path of length \( k \) in \( G \). For \( j \leq k \), let
\[
U_j(P) = \{ i < j \mid \text{tp}(a_i, a_{i+1}) = t \},
\]
and
\[
D_j(P) = \{ i < j \mid \text{tp}(a_i, a_{i+1}) = \hat{t} \}.
\]
Intuitively, \( U_j(P) \) is the set of steps “up” in the path among the first \( j \) steps, and \( D(P) \) is the set of steps “down” among the first \( j \) steps. Then set \( u_j(P) = |U_j(P)| \) and \( d_j(P) = |D_j(P)| \); note that \( u_j(P) + d_j(P) = j \) for all \( j \leq k \).

**Claim 6.** Suppose that \( 1 \leq k \leq 2s + 1 \) and \( P = \langle a_0, \ldots, a_k \rangle \) is a path in \( G \). Let \( u = u_k(P) \) and \( d = d_k(P) \). Then there is \( i < n \) such that
\[
a_k(i - u(s + 1) + ds) < a_0(i) < a_k(i - us + d(s + 1)).
\]

**Remark 7.** Implicit in the statement of the claim is the assertion that
\[
0 \leq i - u(s + 1) + ds < i - us + d(s + 1) < n,
\]
the truth of which will follow readily from the proof.

**Proof of Claim 6.** Define a function \( f : k + 1 \to \mathbb{Z} \) by letting \( f(j) = u_j(P) - d_j(P) \) for every \( j \leq k \). Then \( f \) satisfies the hypotheses of Lemma 4, so, letting \( M = \max(f) \) and \( m = \min(f) \), we have \( M - m \leq k \leq 2s + 1 \).

Let \( i = M(s + 1) \). Note that
\[
M(s + 1) \leq (2s + 1)(s + 1) = 2s^2 + 3s + 1,
\]
so we certainly have \( i < n \).

**Subclaim 8.** For every \( 0 < j \leq k \), we have
\[
a_j(i - sf(j) - u_j(P)) < a_0(i) < a_j(i - sf(j) + d_j(P)).
\]

**Remark 9.** Implicit in the statement of this subclaim is the assertion that, for each \( 0 < j \leq k \), we have
\[
0 \leq i - sf(j) - u_j(P) < i - sf(j) + d_j(P) < n.
\]
This will follow readily from the proof.

**Proof of Subclaim 8.** We proceed by induction on \( j \). We begin by proving the subclaim for \( j = 1 \). Suppose first that \( \text{tp}(a_0, a_1) = t \), so \( f(1) = 1, u_1(P) = 1, \) and \( d_1(P) = 0 \). Then \( M \geq 1 \), so \( i \geq s + 1 \). Therefore, since \( \text{tp}(a_0, a_1) = t \), the preliminary observations at the beginning of the proof of the theorem imply that
\[
a_1(i - s - 1) < a_0(i) < a_1(i - s),
\]
as desired.

If, on the other hand, \( \text{tp}(a_0, a_1) = \hat{t} \), and hence \( f(1) = -1, u_1(P) = 0, \) and \( d_1(P) = 1 \), then \( m \leq -1 \). Therefore, we have \( M \leq 2s \), so \( i = M(s + 1) \leq 2s^2 + 2s < n - s - 1 \). Therefore, since \( \text{tp}(a_0, a_1) = \hat{t} \), the preliminary observations at the beginning of the proof imply that
\[
a_1(i + s) < a_0(i) < a_1(i + s + 1),
\]
as desired.
Now suppose that $0 < j < k$ and we have established that
\[ a_j(i - sf(j) - u_j(P)) < a_0(i) < a_j(i - sf(j) + d_j(P)). \]

We will prove the corresponding statement for $j + 1$. Suppose to begin that $\text{tp}(a_j, a_{j+1}) = t$, so $f(j + 1) = f(j) + 1$, $u_{j+1}(P) = u_j(P) + 1$, and $d_{j+1}(P) = d_j(P)$. In this case, it follows that $f(j) \leq (M - 1)$ and $u_j(P) \leq (M - 1)$. In particular, we have
\[ i - sf(j) - u_j(P) \geq M(s + 1) - s(M - 1) - (M - 1) = s + 1 > s. \]

Therefore, by the preliminary observations, we have
\[ a_{j+1}(i - sf(j) - u_{j+1}(P) - s - 1) < a_j(i - sf(j) - u_j(P)) \]
\[ a_{j+1}(i - sf(j + 1) - u_{j+1}(P)) < a_j(i - sf(j) - u_j(P)) \]

and
\[ a_j(i - sf(j) + d_j(P)) < a_{j+1}(i - sf(j) + d_j(P) - s) \]
\[ a_j(i - sf(j) + d_j(P)) < a_{j+1}(i - sf(j + 1) + d_{j+1}(P)). \]

Combining these inequalities with the inductive hypothesis yields
\[ a_{j+1}(i - sf(j + 1) - u_{j+1}(P)) < a_0(i) < a_{j+1}(i - sf(j + 1) + d_{j+1}(P)), \]
as desired.

On the other hand, suppose that $\text{tp}(a_j, a_{j+1}) = \hat{t}$, so $f(j + 1) = f(j) - 1$, $u_{j+1}(P) = u_j(P)$, and $d_{j+1}(P) = d_j(P) + 1$. In this case, it follows that $f(j) \geq (m + 1)$ and $d_j(P) \leq - (m + 1)$. In particular, we have
\[ i - sf(j) + d_j(P) \leq i - s(m + 1) - (m + 1) = i - (m + 1)(s + 1). \]

We know that $M - m \leq 2s + 1$, so $m + 1 \geq M - 2s$. As a result, the above inequality becomes
\[ i - sf(j) + d_j(P) \leq M(s + 1) - (M - 2s)(s + 1) = 2s^2 + 2s < n - s - 1. \]

Therefore, by the preliminary observations, we have
\[ a_{j+1}(i - sf(j) - u_j(P) + s) < a_j(i - sf(j) - u_j(P)) \]
\[ a_{j+1}(i - sf(j + 1) - u_{j+1}(P)) < a_j(i - sf(j) - u_j(P)) \]

and
\[ a_j(i - sf(j) + d_j(P)) < a_{j+1}(i - sf(j) + d_j(P) + s + 1) \]
\[ a_j(i - sf(j) + d_j(P)) < a_{j+1}(i - sf(j + 1) + d_{j+1}(P)). \]

Combining these inequalities with the inductive hypothesis yields
\[ a_{j+1}(i - sf(j + 1) - u_{j+1}(P)) < a_0(i) < a_{j+1}(i - sf(j + 1) + d_{j+1}(P)), \]
as desired, finishing the proof of the subclaim. \hfill $\Box$

Since $f(k) = u_k(P) - d_k(P)$, we have
\[ i - u(s + 1) + ds = i - sf(k) - u_k(P) \quad \text{and} \quad i - us + d(s + 1) = i - sf(k) + d_k(P). \]

Therefore, the claim follows immediately from Subclaim 8. \hfill $\Box$
Now suppose for sake of contradiction that $G$ has an odd cycle of length $2s + 1$ or shorter. In other words, there is a positive integer $k \leq s$ and a path $C = \langle a_0, \ldots, a_{2k+1} \rangle$ with $a_0 = a_{2k+1}$. Let $u = u_k(C)$ and $d = d_k(C)$. Note that $u + d = 2k + 1$. Apply Claim 6 to find $i < n$ such that

$$a_{2k+1}(i - u(s + 1) + ds) < a_0(i) < a_{2k+1}(i - us + d(s + 1)).$$

Since $a_0 = a_{2k+1}$, this reduces to

$$i - u(s + 1) + ds < i < i - us + d(s + 1).$$

Cancelling $i$ from all three terms yields

$$ds - u(s + 1) < 0 < d(s + 1) - us.$$

Since $d$ and $u$ are both non-negative integers, this implies that they are both nonzero. Therefore, the left inequality gives us

$$\frac{d}{u} < \frac{s + 1}{s},$$

and the right inequality gives us

$$\frac{s}{s + 1} < \frac{d}{u},$$

so we have

$$\frac{s}{s + 1} < \frac{d}{u} < \frac{s + 1}{s}.$$

In particular, $\frac{d}{u}$ is close to 1. But we know that $d + u = 2k + 1$; the assignments of values to $d$ and $u$ subject to this constraint that put $\frac{d}{u}$ closest to 1 are either $d = k$ and $u = k + 1$ or vice versa. But $k \leq s$, so, if $d = k$ and $u = k + 1$, then

$$\frac{d}{u} \leq \frac{s}{s + 1},$$

and, if $d = k + 1$ and $u = k$, then

$$\frac{d}{u} \geq \frac{s + 1}{s}.$$

Either possibility gives us a contradiction, so we are done. \hfill \Box

We end this note by making a few further observations about these disjoint type graphs. We first point out a minor error in the literature. In [1, Remark 1], the authors write, using slightly different terminology, that, for any positive integer $n \geq 3$, the graph $G(\beta, t_n^1)$ has no odd cycles of length less than $2\lceil n/2 \rceil$. This is true for $n = 3$ but false for every larger value of $n$: $G(\beta, t_n^1)$ always has a cycle of length 5, as long as $\beta$ is large enough to allow room for the cycle. In fact, we have the following general result, showing that Theorem 5 is sharp in a sense.

**Proposition 10.** Suppose that $0 < s < n < \omega$ and

$$\beta > (n - 1)(2s + 3) + (2s + 1)(2s + 2).$$

Then the graph $G(\beta, t_n^s)$ has a cycle of length $2s + 3$. 
Proof. Let \( m = 2s + 3 \). We will define a path \( \langle a_0, a_1, \ldots, a_m \rangle \) in \( G(\beta, t^n) \) with \( a_m = a_0 \). First define \( a_0 = a_m \) by letting \( a_m(i) = im \) for all \( i < n \). The definition of each of the remaining elements of the cycle depends on the parity of its index. For \( j \) with \( 0 < j \leq s + 1 \), define \( a_{2j-1} \) by setting
\[
a_{2j-1}(i) = (i + s + j)m - (2j - 1)
\]
for all \( i < n \), and define \( a_{2j} \) by setting
\[
a_{2j}(i) = (i + j)m - 2j
\]
for all \( i < n \). The following facts are easily verified and left to the reader.

- For all \( j \leq s \), \( \text{tp}(a_{2j}, a_{2j+1}) = t^n_s \).
- For all \( j \leq s \), \( \text{tp}(a_{2j+1}, a_{2j+2}) = t^n_s \).
- \( \text{tp}(a_{2s+2}, a_m) = t^n_s \).
- The largest element of any of the vertices in the cycle is
\[
a_{2s+1}(n - 1) = (n - 1)(2s + 3) + (2s + 1)(2s + 2).
\]
Therefore, \( \langle a_0, a_1, \ldots, a_m \rangle \) forms a cycle of length \( 2s + 3 \) in \( G(\beta, t^n) \). \( \square \)

We conclude by noting the following result, which is one of the primary reasons for interest in disjoint type graphs. The result is due to Erdős and Hajnal [2]; the special case \( t = t^3_1 \) is due to Erdős and Rado [4]. A proof of the full result can be found in [1, Theorem 2.1].

**Theorem 11.** Suppose that \( n \) is a positive integer and \( t \) is a disjoint type of width \( n \). For every infinite cardinal \( \kappa \), the graph \( G(\kappa, t) \) has chromatic number \( \kappa \).

**References**


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