VCU

MATH 504

ALGEBRAIC STRUCTURES AND FUNCTIONS

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TEST 3

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Score: 100

Directions. Answer the questions in the space provided. Unless noted otherwise, you must show and explain your work to receive full credit. In proofs, justify each step to the extent reasonable.

This is a closed-book, closed-notes test. Calculators, computers, etc., are not used.

There are 9 numbered questions. Questions 5, 6, and 7 concern rings. All other questions are about groups.

Lagrange's Theorem:
If H is a subgroup of G, then |H| divides |G|.

1. Suppose that $\varphi : \mathbb{Z} \times \mathbb{Z} \to S_9$ is a homomorphism for which $\varphi(1,0) = (3\ 5)(2\ 4)$ and $\varphi(0,1) = (1\ 7)(6\ 8\ 9)$. Find the kernel of φ.

Note that $\varphi(m,n) = \varphi((m,0)+(0,n))$

$= \varphi(m,0)\varphi(0,n) = \varphi(m(1,0))\,\varphi(n(0,1))$

$= \varphi(1,0)^m\,\varphi(0,1)^n = ((3 5)(2 4))^m((1 7)(6 8 9))^n$

$= (3\ 5)^m(2\ 4)^m(1\ 7)^n(6\ 8\ 9)^n$

This can only be i if m is even and n is a multiple of 2 and 3, i.e. 6

Thus $\boxed{\ker\varphi = 2\mathbb{Z} \times 6\mathbb{Z},}$

2. Consider the subgroup $\langle 9\rangle$ of $\mathbb{Z}_{12}$.

(a) List all cosets of $\langle 9\rangle$.

$\langle 9\rangle = \{9, 6, 3, 0\}$. Cosets:

$$\boxed{\begin{aligned} 0+\langle 9\rangle &= \{9, 6, 3, 0\} \\ 1+\langle 9\rangle &= \{10, 7, 4, 1\} \\ 2+\langle 9\rangle &= \{11, 8, 5, 2\} \end{aligned}}$$

(b) ~~Is~~ What familiar group is $\mathbb{Z}_{12}/\langle 9\rangle$ isomorphic to?

$\mathbb{Z}_{12}/\langle 9\rangle = \{0+\langle 9\rangle, 1+\langle 9\rangle, 2+\langle 9\rangle\}$

has just 3 elements, so this group can only be isomorphic to $\boxed{\mathbb{Z}_3}$.

3. Suppose H is a normal subgroup of a group G.
Prove or disprove: If H and G/H are both abelian, then G is also abelian.

This is FALSE

Consider $G = S_3 = \{\rho_0, \rho_1, \rho_2, \mu_1, \mu_2, \mu_3\}$
and .... $H = A_3 = \{\rho_0, \rho_1, \rho_2\}$

Then $H = A_3 \cong \mathbb{Z}_3$ is abelian.
And $G/H \cong \mathbb{Z}_2$ is abelian.

<u>But</u> $G = S_3$ is <u>not</u> <u>abelian</u>

4. Suppose $\varphi : G \to H$ is a homomorphism and $|G|$ is prime.
Show that φ is either the trivial homomorphism or is one-to-one.
(The *trivial homomorphism* $\varphi : G \to H$ is $\varphi(x) = e_H$ for all $x \in G$.)

Proof Suppose $\varphi : G \to H$ is a homomorphism and $|G|$ is prime.
We know $\ker \varphi \leq G$, and by Lagrange's theorem $|\ker \varphi|$ divides the prime number $|G|$.
Thus either $|\ker \varphi| = 1$ or $|\ker \varphi| = |G|$.

Case I: $|\ker \varphi| = 1$. Then $\ker \varphi = \{e\}$.
Then φ is one-to-one as follows:
Suppose $\varphi(a) = \varphi(b)$. Then $\varphi(a)\varphi(b)^{-1} = e$,
that is, $\varphi(a)\varphi(b^{-1}) = e$, so
$\varphi(ab^{-1}) = e$. Then $ab^{-1} \in \ker \varphi = \{e\}$,
so $ab^{-1} = e$, and $a = b$. Thus φ is one-to-one

Case II Suppose $|\ker \varphi| = |G|$. Then $\ker \varphi = G$, which means $\varphi(x) = e$ for any $x \in G$. Thus φ is the trivial homomorphism. ∎

5. State the definition of a ring.

A ring is a set R on which are defined two binary operations $+$ and $\cdot$, satisfying:

$\mathcal{R}_1$: $\langle R, + \rangle$ is an abelian group with identity 0.

$\mathcal{R}_2$: Multiplication is associative: $a(bc) = (ab)c$ for all $a, b, c \in R$.

$\mathcal{R}_3$: $a(b+c) = ab + ac$ and $(b+c)a = ba + ca$ for all $a, b, c \in R$.

6. List all zero divisors of the ring $\mathbb{Z}_3 \times \mathbb{Z}_5$.

$(1,0)$ $(0,1)$
$(2,0)$ $(0,2)$
$(0,3)$
$(0,4)$

Any one on left times any one on right gives $(0,0)$

7. Show that if a is an element of a ring R, then $0a = 0$.

Note that $0a = (0+0)a = 0a + 0a$.

From $0a = 0a + 0a$ we get

$$0a - 0a = 0a + 0a - 0a$$

$$0 = 0a + 0$$

$$\boxed{0 = 0a}$$

8. Let G be a group with $|G| = pq$, where p and q are prime numbers. Show that every proper subgroup H of G is cyclic. (Recall that *proper* means $H \neq \{e\}$ and $H \neq G$.)

Proof Suppose $|G| = pq$, where p and q are prime. Let H be a proper subgroup of G. Then $1 < |H| < pq$ and $|H|$ divides $|G| = pq$ by Lagrange's theorem. As p and q are prime, it follows that $|H| = p$ or $|H| = q$. Either way $|H|$ is prime.

Now take an element $a \neq e$, with $a \in H$ and consider the subgroup $\langle a \rangle \leq H$. Again by Lagrange's Theorem, $|\langle a \rangle|$ divides the prime number $|H|$, so it must be that $|\langle a \rangle| = |H|$. Consequently $H = \langle a \rangle$, which means that H is cyclic. $\blacksquare$

9. Suppose $\varphi : G \to H$ is a homomorphism.
Recall that $\varphi[G] = \{\varphi(x) \mid x \in G\} \subseteq H$.
Prove that $\varphi[G]$ is a subgroup of H.

(a) Note $e_H \in \varphi[G]$ because

$$e_H = \varphi(e_G) \in \varphi[G].$$

(b) Next observe that $\varphi[G]$ is closed under multiplication: Suppose $x, y \in \varphi[G]$, which means $x = \varphi(a)$ and $y = \varphi(b)$ for some $a, b \in G$. Then

$$xy = \varphi(a)\varphi(b) = \varphi(ab) \in \varphi[G]$$

(c) Finally suppose $x \in \varphi[G]$, so $x = \varphi(a)$ for some $a \in G$. We need to show $x^{-1} \in \varphi[G]$.

Note $x^{-1} = \varphi(a)^{-1} = \varphi(a^{-1}) \in \varphi[G]$.

The above considerations (a), (b), (c) show that $\varphi[G]$ is a subgroup of H. ∎