VCU

MATH 504

ALGEBRAIC STRUCTURES AND FUNCTIONS

R. Hammack

TEST 2

☞

March 27, 2014

Name: Richard

Score: ____________

Directions. Answer the questions in the space provided. Unless noted otherwise, you must show and explain your work to receive full credit. In proofs, justify each step to the extent reasonable.

This is a closed-book, closed-notes test. Calculators, computers, etc., are not used.

There are 9 numbered questions.

1. (10 pts.) Consider the subset $H = \{(x, 3x) \mid x \in \mathbb{Z}_5\} \subseteq \mathbb{Z}_5 \times \mathbb{Z}_5$.
Show that H is a subgroup of $\mathbb{Z}_5 \times \mathbb{Z}_5$.

ⓐ First, note that $(0,0) = (0, 3\cdot 0) \in H$, so H does contain the identity.

ⓑ Now let's see if H is closed. Take two arbitrary elements $(x, 3x)$ and $(y, 3y)$ in H. Then $(x, 3x) + (y, 3y) = (x+y,\ 3x+3y) = (x+y,\ 3(x+y)) \in H$ so H is indeed closed.

ⓒ Now suppose $(x, 3x) \in H$. The inverse of this element is $(-x, -3x) = (-x, 3(-x)) \in H$, so H contains the inverse of any of its elements.

ⓐ-ⓑ above imply that H is a subgroup

2. (10 pts.) Consider the set $H = \{\sigma \in S_5 \mid \sigma(4) = 3\}$.

(a) $|H| = 4! = 4 \cdot 3 \cdot 2 = \boxed{24}$

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ \square & \square & \square & 3 & \square \end{array}$$

Reason: In making a σ in H there are 4 choices for $\sigma(1)$, then 3 for $\sigma(2)$, then 2 for $\sigma(3)$ and one for $\sigma(5)$

(b) Is H a subgroup of S_5? Why or why not?

No Notice that the identity $\dot{\imath}$ does not satisfy $\dot{\imath}(4) = 3$, so $\dot{\imath} \notin H$. Thus H can't be a subgroup.

3. (14 pts.) Answer the following questions about the given permutations.

(a) Write as a product of disjoint cycles: $(4\,8\,3\,2)(8\,3) =$

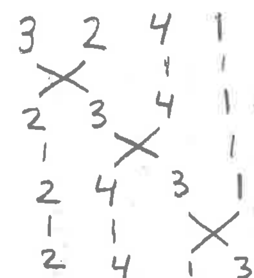
$\boxed{(8\ 2\ 4)}$

(b) Write as a product of disjoint cycles: $\left[(4\,8\,3\,2)(8\,3)\right]^{-1} =$ $(8\,2\,4)^{-1}$

$= \boxed{(4\ 2\ 8)}$

(c) Write as a product of disjoint cycles: $(4\,8\,3\,2\,1\,5)^3 =$ $\boxed{(4\ 2)(8\ 1)(3\ 5)}$

(d) Write $(3\,2\,4\,1)$ as a product of transpositions.



$(3\ 2\ 4\ 1) = \boxed{(1\ 3)(3\ 4)(3\ 2)}$

(e) Is the permutation $(3\,2\,4\,1)$ from part (d) above even or odd?

$\boxed{\text{odd}}$ (It's a product of an odd # of transpositions)

(f) Find the order of the permutation $(1\,2)(4\,6\,5)(3\,9\,8\,7)$.

$\operatorname{lcm}(2, 3, 4) = \boxed{12}$

(g) How many elements does the alternating group A_4 have?

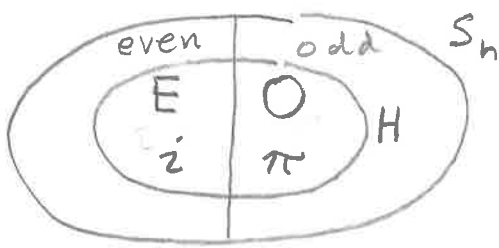
$$|A_4| = \frac{|S_4|}{2} = \frac{4!}{2} = \boxed{12}$$

4. (12 pts.) Prove that if H is a subgroup of S_n, then either all permutations in H are even, or half are even and half are odd.

Proof (Direct) Suppose $H \leq S_n$. Then each element of H is a permutation of $\{1, 2, \cdots, n\}$, so any element is either even or odd. We break into two cases:

Case I Suppose all permutations in H is even. Then we are done!

Case II Suppose H contains at least one odd permutation π. H also contains the (even) identity permutation i.



Let $H = E \cup O$, where E is the set of even permutations in H and O is the set of odd permutations in H. Thus $i \in E$ and $\pi \in H$.

We need to show that half the permutations in H are even and the other half are odd, i.e. $|E| = |O|$.

To show that $|E| = |O|$ it suffices to produce a bijection $\varphi: E \longrightarrow O$

Define φ as $\varphi(\sigma) = \pi\sigma$ where π is a fixed element in O. Notice that this makes sense: If $\sigma \in E$, then $\varphi(\sigma) = \pi\sigma$ is odd because it's a product of an even and an odd permutation. Moreover because H is closed and $\pi, \sigma \in H$ we have $\pi\sigma \in H$, hence $\varphi(\sigma) = \pi\sigma \in O$. Therefore we really do have a function $\varphi: E \to O$.

Note that φ is injective: If $\varphi(\sigma) = \varphi(\mu)$, then $\pi\sigma = \pi\mu$ and $\sigma = \mu$ by cancellation.

Also φ is surjective because given any $\mu \in O$, we have $\pi^{-1}\mu \in E$ (As odd $\cdot$ odd $=$ even) and $\varphi(\pi^{-1}\mu) = \pi\pi^{-1}\mu$ $= \mu$. This shows φ is surjective

Thus we have a bijection $\varphi: E \to O$, so $|E| = |O|$, meaning half the permutations in H are even and the other half are odd.

The above two cases show that <u>either</u> all permutations in H are even, or half are even and half are odd. $\blacksquare$

5. (10 pts.) Give an example of an abelian subgroup H of a non-abelian group G. (That is, give an explicit example of such a G and H. Do not pick H = {e}.)

Example: $A_3 \leq S_3$

$\{ i, (123), (132) \} = \langle (123) \rangle$ is abelian

Non-abelian

6. (10 pts.) Find all abelian groups with 225 elements. $225 = 3^2 5^2$

$\mathbb{Z}_9 \times \mathbb{Z}_{25}$

$\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{25}$

$\mathbb{Z}_9 \times \mathbb{Z}_5 \times \mathbb{Z}_5$

$\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_5$

7. (10 pts.) Consider the group S_{10}. State an element of this group that has the greatest possible order.

$(1\,2)(3\,4\,5)(6\,7\,8\,9\,10)$

has order $2 \cdot 3 \cdot 5 = 30$

8. (12 pts.) Prove that if G is finite and the only subgroups of G are $\{e\}$ and G, then $|G|$ is prime. (Suggestion: Consider using Lagrange's Theorem.)

Proof (Contradiction) Suppose the only subgroups of G are G and $\{e\}$, and $|G|$ is not prime.

First note that G is cyclic: Take any $a \in G$ with $a \neq e$ and consider $\langle a \rangle$. Since this subgroup contains at least e and a, we have $\langle a \rangle \neq \{e\}$, so it must be that $\langle a \rangle = G$, which means G is cyclic.

Since $|G|$ is not prime, then $|G| = mn$ for $m, n \in \mathbb{Z}$, $m, n > 1$.

Then $G = \langle a \rangle = \{a^1, a^2, a^3, \cdots a^{mn}\}$.

Now let $H = \langle a^n \rangle = \{a^n, a^{2n}, a^{3n}, a^{4n}, \dots, a^{mn}\}$.

Then $1 < m = |H| < mn = |G|$.

Thus H is a subgroup that is neither $\{e\}$ nor G. This contradicts our initial assumption ∎

9. (12 pts.) Show that any cyclic group is abelian.

Proof Suppose G is cyclic. This means there is an element $a \in G$ for which $G = \langle a \rangle = \{a^k \mid k \in \mathbb{Z}\}$.

Now take any two elements $x, y \in G$. Because $G = \langle a \rangle$, we know $x = a^k$ and $y = a^\ell$ for some $k, \ell \in \mathbb{Z}$. Consequently

$$
\begin{aligned}
xy = a^k a^\ell &= a^{k+\ell} \\
&= a^{\ell+k} \\
&= a^\ell a^k \\
&= yx.
\end{aligned}
$$

As we have shown that $xy = yx$ for all $x, y \in G$, it follows that G is abelian. ∎