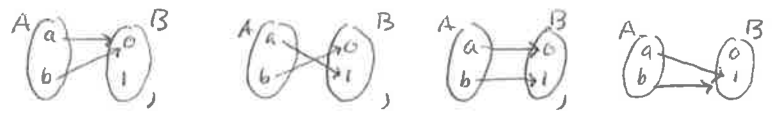
On the first homework assignment most of the class struggled with §0, #18. Many did not attempt it. We can expect lots of proof-type problems like this one, so we'll devote some class time to its solution.

In this problem A is an arbitrary set, $B = \{0,1\}$ and B^A is the set of all functions $f : A \to B$. We are asked to show $|B^A| = |\mathcal{P}(A)|$. To accomplish this we must find a one-to-one and onto function $\varphi : B^A \longrightarrow \mathcal{P}(A)$.

To get an idea of how to proceed, look at the example where $A = \{a, b\}$. Let's list out B^A and $\mathcal{P}(A)$.

$B^A = \{$  $\}$

φ

$\mathcal{P}(A) = \{ \ \emptyset, \ \ \{a\}, \ \ \{b\}, \ \ \{a, b\} \ \}$

Notice that indeed $|B^A| = 4 = |\mathcal{P}(A)|$. But also the example suggests an actual function $\varphi : B^A \longrightarrow \mathcal{P}(A)$ sending any function $f \in B^A$ to the set $\{x \in A \mid f(x) = 1\}$ in $\mathcal{P}(A)$, (just below f). Now that we've got an idea, let's put it all together.

<u>§0 #18</u> Suppose A is any set (finite or infinite) and B^A is the set of all functions $f : A \to B$. Prove that $|B^A| = |\mathcal{P}(A)|$.

<u>Proof</u> We need to produce a one-to-one and onto function $\varphi : B^A \to \mathcal{P}(A)$. To do this, let $\varphi : B^A \to \mathcal{P}(B)$ be the function defined as $\varphi(f) = \{x \in A \mid f(x) = 1\}$.

<u>First note that φ is one-to-one</u>: Suppose $f, g \in B^A$ and $f \neq g$. We want to show $\varphi(f) \neq \varphi(b)$. Now, $f \neq g$ means that there is an element $a \in A$ for which $f(a) \neq g(a)$. Then either $f(a) = 0$ and $g(a) = 1$, <u>or</u> $f(a) = 1$ and $g(a) = 0$. Let's say $f(a) = 0$ and $g(a) = 1$. (The other case is nearly identical.) Now since $f(a) = 0$, we know $a \notin \{x \in A \mid f(x) = 1\} = \varphi(f)$. And because $g(a) = 1$, we know $a \in \{x \in A \mid g(x) = 1\} = \varphi(g)$. Therefore $a \notin \varphi(f)$ but $a \in \varphi(g)$, so $\varphi(f) \neq \varphi(g)$.

<u>Next note that φ is onto</u>: Take an arbitrary set $X \in \mathcal{P}(A)$, which is to say X is an arbitrary subset $X \subseteq A$. Now construct a function $f : A \to B$ defined as $f(x) = \begin{cases} 0 & \text{if } x \notin X \\ 1 & \text{if } x \in X \end{cases}$

Therefore $f \in B^A$ and note that it has been constructed so that $\varphi(f) = \{x \in A \mid f(x) = 1\} = X$. Therefore φ is onto. $\blacksquare$