Section 7 Generating Sets and Cayley Digraphs

- The goal of this section is to generalize the idea of a subgroup generated by $a \in G$.

Recall: If $a \in G$, then $\langle a \rangle = \{ \dots a^{-2}, a^{-1}, a^0, a^1, a^2, \dots \} \leq G$.

Definition: If $a, b \in G$, then $\langle a, b \rangle = \{ \dots a^5 b^{-3}, ab^2a^4b, a^0b^0, ab, ba, \dots \}$
$= \{$ all products of powers of a and $b \} \subseteq G$.

Observation: $\boxed{\begin{array}{l} H = \langle a,b \rangle \text{ is a subgroup of } G \\ 1.\ \ \langle a,b \rangle \text{ is closed} \\ 2.\ \ e = a^0 b^0 \in H \\ 3.\ \ \text{Inverse of } ab^2a^4b \text{ is } b^{-1}a^{-4}b^{-2}a^{-1}\text{, etc.} \end{array}}$

$H = \langle a,b \rangle$ is called the subgroup of G generated by a & b.

- If G is abelian, then $\langle a,b \rangle = \{ a^m b^n \mid m,n \in \mathbb{Z} \}$
- If G is additive, then $\langle a,b \rangle = \{ ma + nb \mid m,n \in \mathbb{Z} \}$
- $\langle a_1, a_2, a_3 \dots a_k \rangle = \{$ all products of powers of $a_1, a_2, \dots a_k \} \leq G$.

Ex $\langle 4,6 \rangle \subseteq \mathbb{Z}_{12}$ $\quad \langle 4,6 \rangle = \{0, 4, 8, 6, 10, 2\} \leq \langle 2 \rangle \leq \mathbb{Z}_{12}$

Ex $\langle 2,3 \rangle \subseteq \mathbb{Z}_{12}$ $\quad \langle 2,3 \rangle = \{2, 3, 1, \dots \} = \langle 1 \rangle = \mathbb{Z}_{12}$

Ex $\langle 3,6,9 \rangle \subseteq \mathbb{Z}_{18}$ $\quad \langle 3,6,9 \rangle = \{0, 3, 6, 9, 12, 15\} = \langle 3 \rangle \leq \mathbb{Z}_{12}$

Ex $\langle a \rangle \leq V$ $\quad \langle a \rangle = \{e, a\}$

Ex $\langle a,b \rangle \leq V$ $\quad \langle a,b \rangle = \{e, a, b, c\} = V$

Definition If $G = \langle a_1, a_2, a_3, \dots a_k \rangle$ we say that the set $\{a_1, a_2, \dots a_k\}$ generates G.

Definition If $G = \langle a_1, a_2, a_3, \dots a_k \rangle$ we say G is finitely generated. The generators are $a_1\ a_2 \dots a_k$.

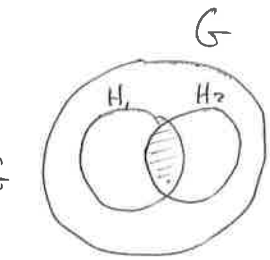
Ex $V = \langle a, b \rangle$ V is finitely generated.

$\mathbb{Z} = \langle 1 \rangle$ $\mathbb{Z}$ is finitely generated

$$\left.\begin{array}{l} \mathbb{Z} \times \mathbb{Z} = \langle (1,0), (0,1) \rangle \\ \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} = \langle (1,0,0), (0,1,0), (0,0,1) \rangle \\ \mathbb{Z}_8 = \langle 3 \rangle \end{array}\right\} \text{all finitely generated.}$$

But $\mathbb{R}$ (i.e. $\langle \mathbb{R}, + \rangle$) is not finitely generated.

If Time permits



Text Shows: If $H_1, H_2 \leq G$ then $H_1 \cap H_2 \leq G$

Example $\langle 2 \rangle = \{0, 2, 4, 6, 8, 10\} \leq \mathbb{Z}_{12}$

$\langle 3 \rangle = \{0, 3, 6, 9\} \leq \mathbb{Z}_{12}$

$\langle 2 \rangle \cap \langle 3 \rangle = \{0, 6\} = \langle 6 \rangle \leq \mathbb{Z}_{12}$

Further $H_1 \cap H_2 \cap H_3 \leq G$.

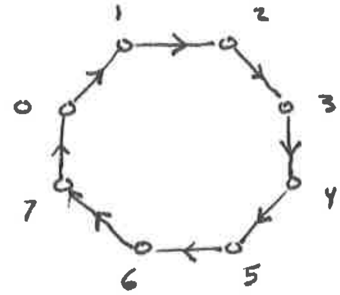
Theorem If $H_i \leq G$ for all $i \in I$ then $\bigcap_{i \in I} H_i \leq G$.

Consequence $\langle a_1\, a_2 \dots a_k \rangle$ is smallest subgroup of G containing $\{a_1, a_2, \dots a_k\}$

Given a set of generators of a group, we form what is called a Cayley digraph. Rather than give a definition (which you may read in the text) let's illustrate by example.

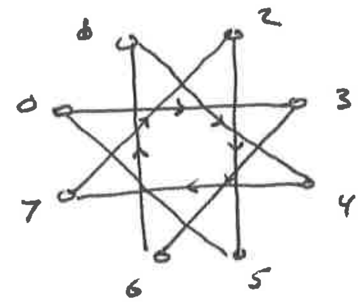
Consider generator $\{1\}$ of $\mathbb{Z}_8$

a o———→———o b means $a +_8 1 = b$



Consider generator $\{3\}$ of $\mathbb{Z}_8$

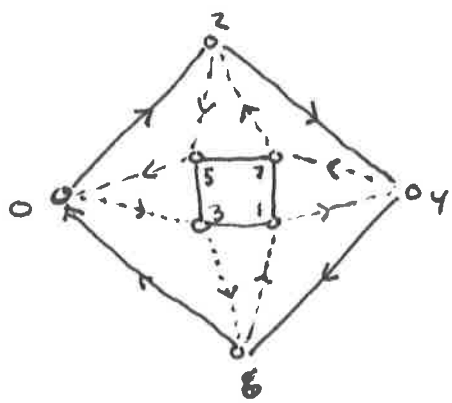
a o———→———o b means $a +_8 3 = b$



Consider generators $\{2, 3\}$ of $\mathbb{Z}_8$

a o———→———o b means $a +_8 2 = b$

o - - - → - - - o means $a +_8 3 = b$



<u>Ex</u> $G = \{000, 001\ 010\ 011\ 100\ 101\ 110\ 111\}$ Klein 8-group

Generators: $\{100, 010, 001\}$

a o—————————o b means $a + 100 = b$

a o — — — — o b means $a + 010 = b$

o · · · · · · · · o means $a + 001 = b$