# Section 3 Isomorphic Binary Structures

**Definition** If set $S$ has a binary operation $*$, we denote this as $\langle S, * \rangle$ and call this a binary structure.

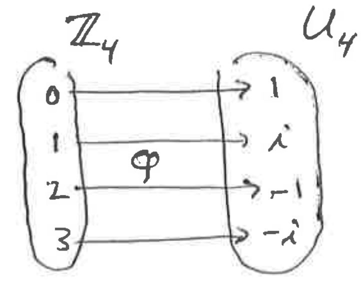
**Examples** $\langle \mathbb{Z}_4, +_4 \rangle$ $\langle U_4 \cdot \rangle$

| $+_4$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 1 | 2 | 3 | 0 |
| 2 | 2 | 3 | 0 | 1 |
| 3 | 3 | 0 | 1 | 2 |

| $\cdot$ | $1$ | $i$ | $-1$ | $-i$ |
|---|---|---|---|---|
| $1$ | $1$ | $i$ | $-1$ | $-i$ |
| $i$ | $i$ | $-1$ | $-i$ | $1$ |
| $-1$ | $-1$ | $-i$ | $1$ | $i$ |
| $-i$ | $-i$ | $1$ | $i$ | $-1$ |

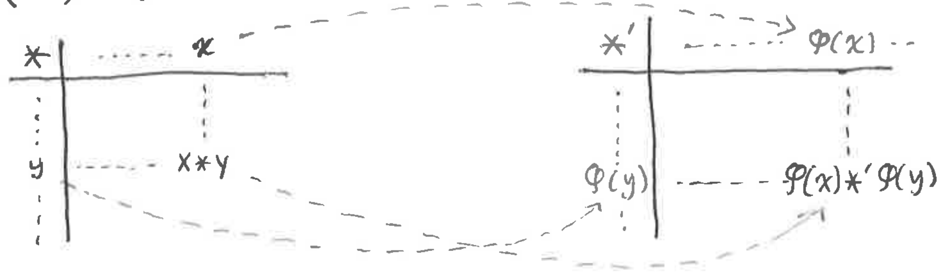
Earlier we noted that these two things have essentially the same structure. Only the names of the symbols differ, otherwise the elements combine in the same way. We call such identical structures isomorphic. How could this idea be formalized?

To answer this, note that for the above example, there is a function $\varphi : \mathbb{Z}_4 \to U_4$ that "renames" the elements




But this renaming happens in a way that respects the algebraic structure. To better understand this, imagine we have two binary structures that are isomorphic.

$\langle S, * \rangle$ $\langle S', *' \rangle$




<u>Note</u>: For these to have the same structure, it must be true that $\varphi(x * y) = \varphi(x) *' \varphi(y)$.

**Definition** $\langle S, * \rangle$ and $\langle S', *' \rangle$ are said to be **isomorphic** binary structures if there is a bijection $\varphi : S \to S'$ satisfying $\varphi(x * y) = \varphi(x) *' \varphi(y)$ for all $x, y \in S$.

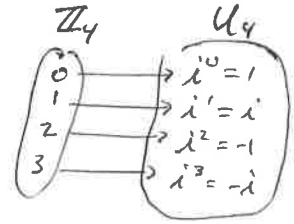
note typo in text.

This is denoted $\langle S, * \rangle \cong \langle S', *' \rangle$, and $\varphi$ is called an **isomorphism**

**Example** $\langle \mathbb{Z}_4, +_4 \rangle \cong \langle U_4, \cdot \rangle$

Reason: $\varphi : \mathbb{Z}_4 \to U_4$

$\varphi(x) = i^x$



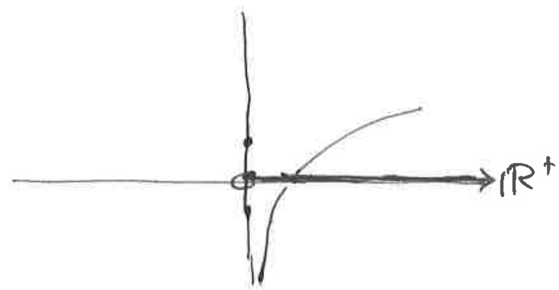

it's a bijection.

$$\varphi(x +_4 y) = i^{x +_4 y} = i^{x + y \ (\mathrm{mod}\ 4)} = i^x \cdot i^y = \varphi(x) \cdot \varphi(y)$$

**Example** $\langle \mathbb{R}^+, \cdot \rangle \cong \langle \mathbb{R}, + \rangle$

Reason: $\ln : \mathbb{R}^+ \to \mathbb{R}$ is bijection

$\ln(xy) = \ln(x) + \ln(y)$




**Example** $P = \{ \cdots \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8, 16, \cdots \} = \{ 2^n \mid n \in \mathbb{Z} \} \subseteq \mathbb{Q}$

$$\langle \mathbb{Z}, + \rangle \cong \langle P, \cdot \rangle$$

$\mathbb{Z}$ $\quad 3 + 2 = 5$

$P$ $\quad 2^3 \cdot 2^2 = 2^5$

$\quad 8 \cdot 4 = 32$

Reason: $\varphi : \mathbb{Z} \to P$

$\varphi(n) = 2^n$

This is a bijection: • $n \neq m \Rightarrow 2^m \neq 2^n \Rightarrow \varphi(m) \neq \varphi(n)$

• given $2^n \in P$, $\varphi(n) = 2^n$.

$$\varphi(m+n) = 2^{m+n} = 2^m \cdot 2^n = \varphi(m) \cdot \varphi(n)$$

**Example**: $\langle \mathbb{Z}_n, +_n \rangle \cong \langle U_n, \cdot \rangle$

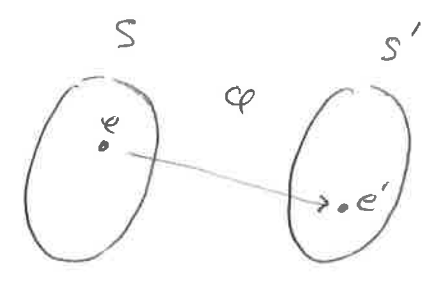
**Theorem**

Suppose $\langle S, * \rangle$ has identity $e$

and $\langle S', *' \rangle$ has identity $e'$

and $\varphi : S \to S'$ is an isomorphism

Then $\varphi(e) = e'$




**Ex** $\ln : \mathbb{R}^+ \longrightarrow \mathbb{R}$ $\qquad \ln(1) = 0$

identity 1 $\qquad$ identity 0

**Ex** $\varphi : \mathbb{Z}_4 \longrightarrow U_4$ $\qquad \varphi(0) = 1$

identity 0 $\qquad$ identity 1

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How can we tell if two binary structures are not isomorphic?

**Ex** $\langle \mathbb{Z}_4, +_4 \rangle \not\cong \langle \mathbb{Z}_5, +_5 \rangle$ because no 1-1, onto $\varphi : \mathbb{Z}_4 \to \mathbb{Z}_5$

**Ex** $\langle \mathbb{Q}, \cdot \rangle \not\cong \langle \mathbb{R}, \cdot \rangle$ because no 1-1 onto $\varphi : \mathbb{Q} \to \mathbb{R}$

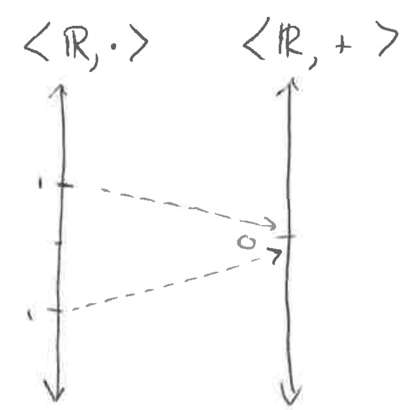
$(\,|\mathbb{Q}| \neq |\mathbb{R}|\,)$

**Ex** How do $\langle \mathbb{R}, \cdot \rangle$ and $\langle \mathbb{R}, + \rangle$ compare?

Could they be isomorphic?

What if there were an isomorphism

$\varphi : \mathbb{R} \longrightarrow \mathbb{R}$




$\varphi(1) = 0$

$\varphi((-1)(-1)) = 0$

$\varphi(-1) + \varphi(-1) = 0$

$2\varphi(-1) = 0$

$\varphi(-1) = 0$ $\qquad$ but then $\varphi$ can't be 1-1. $\qquad$ Not an isomorphism