Section 2 Binary Operations

The familiar algebraic operations of $+ \cdot \div$ and $-$ can be thought of as functions that take two arguments and spit out an answer. $3+5=8$ is like $+(3,5)=8$. The purpose of this section is to generalize and formalize this idea.

<u>Definition</u>: A binary operation $*$ on a set S is a function $*: S \times S \to S$. The element $*(a,b)$ is written $a*b$.

<u>Example</u> $+: \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ $\quad +(5,3) = 5+3 = 8$

<u>Example</u> $+_4 : \mathbb{Z}_4 \times \mathbb{Z}_4 \longrightarrow \mathbb{Z}_4$ $\quad +_4(2,2)=0 \quad +_4(3,3)=2$

$+_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

<u>Example</u> $S = \{$○, □, ✂$\}$ $\quad *: S \times S \to S$ is as follows

$*$	○	□	✂
○	○	□	○
□	□	□	✂
✂	○	✂	✂

○ $*$ □ = □

○ $*$ ✂ = ○

<u>Example</u> $\div : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ $\quad \div(x,y) = x \div y$

This is <u>not</u> a binary relation because it is not a well-defined function. $\div(5,0)$ has no value

However $\div : \mathbb{R}^* \times \mathbb{R}^* \to \mathbb{R}^*$ <u>is</u> a binary operation.

<u>Example</u> $\cap : \mathcal{P}(A) \times \mathcal{P}(A) \longrightarrow \mathcal{P}(A)$

Definition Binary operation $*$ is <u>commutative</u> if $a*b = b*a \ \forall \ a, b \in S$.

Every binary operation we've looked at today is commutative. Here's one that isn't.

<u>Ex</u> $S = \{ f \mid f: \mathbb{R} \to \mathbb{R} \text{ is a function} \}$

Consider binary operation of composition $\circ : S \times S \to S$

$f(x) = \sqrt[3]{x}$ $\quad$ $f \circ g(x) = f(g(x)) = \sqrt[3]{x+1}$

$g(x) = x+1$ $\quad$ $g \circ f(x) = g(f(x)) \quad \sqrt[3]{x} + 1$

Thus $f \circ g \neq g \circ f$

Definition Binary operation $*$ is <u>associative</u> if $a*(b*c) = (a*b)*c$ for all $a, b, c \in S$.

<u>Ex</u> $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is associative $a+(b+c) = (a+b)+c$

Most of the examples we've seen today are associative.

One is not can you find it?

$\bigcirc * (✂ * \square) = \bigcirc * ✂ = \bigcirc$

$(\bigcirc * ✂) * \square = \bigcirc * \square = \square$

Thus $\bigcirc * (✂ * \square) \neq (\bigcirc * ✂) * \square$

Definition ~~Binary operation~~ Suppose S has binary operation $*$. An element $e \in S$ is an <u>identity</u> if $e*a = a = a*e \ \forall \ a \in S$

<u>Ex</u> $+$ on $\mathbb{Z}$ has identity 0: $a+0 = a = 0+a \quad \forall a \in S$.

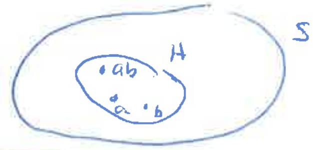
<u>Ex</u> $\cdot$ on $\mathbb{Z}$ has identity 1: $a \cdot 1 = a = 1 \cdot a \quad \forall a \in S$

<u>Ex</u> Rock-paper-sissors has no identity.

<u>Theorem</u> If S has an identity e, then it has only one identity

<u>Proof</u> Suppose e and e' are identities. Then $e = ee' = e'$.

Definition Given $* : S \times S \to S$, suppose $H \subseteq S$. H is closed under $*$ if $\forall\, a, b \in H$, $a * b \in H$.



Ex $\mathbb{Q}^* \subseteq \mathbb{R}^*$. $\mathbb{Q}^*$ is closed under the operation $\div$

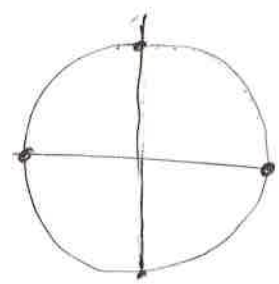
$\frac{a}{b}, \frac{c}{d} \in \mathbb{Q}^*$ and $\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc} \in \mathbb{Q}^*$

Ex $\mathbb{Z}^* \subseteq \mathbb{R}^*$. $\mathbb{Z}^*$ not closed under $\div$.

$5 \in \mathbb{Z}^*$, $3 \in \mathbb{Z}^*$ but $5 \div 3 = \frac{5}{3} \notin \mathbb{Z}^*$.

Ex $H = \{+1, -1\} \subseteq U_4 = \{1, i, -1, -i\}$

H is closed under multiplication



$U_4 \subseteq U$ is closed under mult

$U \subseteq \mathbb{C}$ is closed under mult.

Ex $P = \{2^n \mid n \in \mathbb{Z}\} = \{\ldots \frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8, 16, 32 \ldots\}$

$P \subseteq \mathbb{R}$ is closed under $\cdot$

Proof $2^k, 2^m \in P$ $\quad 2^k \cdot 2^m = 2^{k+m} \in P$.