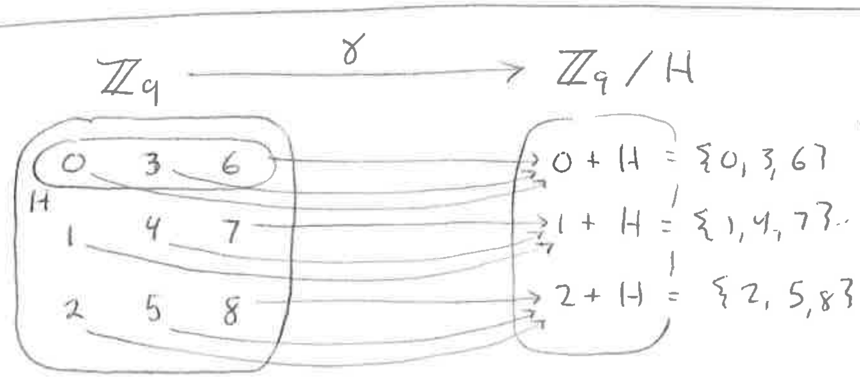
The Fundamental Homomorphism Theorem

Theorem Suppose $H \leq G$ is normal. Then $\gamma : G \longrightarrow G/H$ defined as $\gamma(x) = xH$ is a homomorphism, and $\ker(\gamma) = H$.

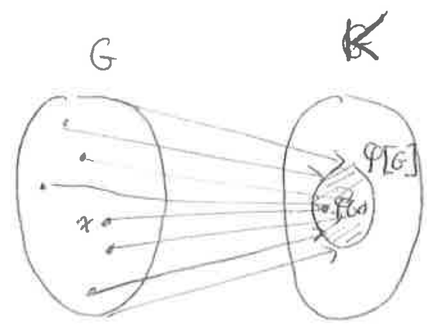
Proof: $\gamma(xy) = xyH = xH\,yH = \gamma(x)\gamma(y)$.
$\ker(\gamma) = \{x \in G \mid \gamma(x) = H\} = \{x \in G \mid xH = H\} = H$.

Example



γ sends each element to the coset it's in

Recall:
Given homomorphism $\varphi : G \longrightarrow K$,
$\varphi[G] = \{\varphi(x) \mid x \in G\} \leq K$.
φ is onto $\iff \varphi[G] = K$.



Theorem Fundamental Homomorphism Theorem
Suppose $\varphi : G \longrightarrow K$ is a homomorphism with $H = \ker(\varphi)$.
Then $\mu : G/H \longrightarrow \varphi[G]$ defined as $\mu(xH) = \varphi(x)$
is an isomorphism satisfying $\mu(\gamma(x)) = \varphi(x)$.

$$\begin{tikzcd}
G \arrow[rr, "\varphi"] \arrow[dr, "\gamma"'] & & K \\
& G/H \arrow[ur, "\mu"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
x \arrow[rr, mapsto, "\varphi"] \arrow[dr, mapsto, "\gamma"'] & & \varphi(x) \\
& xH \arrow[ur, mapsto, "\mu"'] &
\end{tikzcd}$$

Proof

$\mu(xH\,yH) = \mu(xyH) = \varphi(xy) = \varphi(x)\varphi(y) = \mu(xH)\mu(xy)$

μ is onto: Suppose $y \in \varphi[G]$, so $y = \varphi(x)$ for some $x \in G$.
Then $y = \varphi(x) = \mu(xH)$, so μ is onto.

μ is 1-1: $\ker(\mu) = \{aH \in G/H \mid \mu(aH) = e_K\}$

$= \{aH \in G/H \mid \varphi(a) = e_K\}$

$= \{aH \in G/H \mid a \in \ker(\varphi)\}$

$= \{aH \in G/H \mid a \in H\} = \{eH\}$

Since $\ker(\mu)$ is trivial, μ is 1-1.

Usually The F.H.T is applied to situations where φ is onto. In such cases it becomes:

Theorem F.H.T

Suppose $\varphi: G \to K$ is an onto homomorphism and $H = \ker(\varphi)$. Then $\mu: G/H \to G'$ defined as $\mu(xH) = \varphi(x)$ is an isomorphism satisfying $\mu(\gamma(x)) = \varphi(x)$.

$$\begin{tikzcd} G \arrow[rr, "\varphi"] \arrow[dr, "\gamma"'] & & K \\ & G/H \arrow[ur, "\mu", "\cong"'] & \end{tikzcd}$$

$\mathbb{Z} \longrightarrow \mathbb{Z}_n$

K