Section 13 Homomorphisms

An isomorphism $\varphi: G \to K$ is a bijection satisfying $\varphi(ab) = \varphi(a)\varphi(b)$. $\forall\, a, b \in G$. In words, it respects the algebraic structure of G. Some maps are like isomorphisms, but they are not bijective. Such maps are called homomorphisms.

Definition A map $\varphi: G \to K$ is a homomorphism if $\varphi(ab) = \varphi(a)\varphi(b)$ $\forall\, a, b \in G$.

Examples:

1. Any isomorphism is a homomorphism.
2. Trivial homomorphism $\varphi: G \to K$ $\varphi(g) = e$.
3. Any linear map $T: \mathbb{R}^n \to \mathbb{R}^m$. $T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y})$
4. $\det: GL(n, \mathbb{R}) \to \mathbb{R}^*$ $\det(AB) = \det(A)\det(B)$.
5. $\ln: \mathbb{R}^+ \to \mathbb{R}$ $\ln(ab) = \ln(a) + \ln(b)$
6. $|\ |: \mathbb{R}^* \to \mathbb{R}^*$ $|ab| = |a|\,|b|$
7. $\varphi: \mathbb{Z} \to \mathbb{Z}_n$ $\varphi(x) = x \pmod n$ $\varphi(x+y) = (x+y) \bmod n = x \pmod n + y \pmod n$
8. Let $F = \{ f: \mathbb{R} \to \mathbb{R} \mid f \text{ is a function} \}$ Let $a \in \mathbb{R}$

   $\varphi_a: F \to \mathbb{R}$ defined as $\varphi_a(f) = f(a)$.

   Homomorphism: $\varphi_a(f+g) = (f+g)(a) = f(a) + g(a) = \varphi_a(f) + \varphi_a(g)$

   Called the evaluation homomorphism
9. $\mathcal{P} = \{ f: \mathbb{R} \to \mathbb{R} \mid f \text{ is a polynomial} \}$

   Homomorphism $\frac{d}{dx}: \mathcal{P} \to \mathcal{P}$. $\frac{d}{dx}(f) = f'$

   $\frac{d}{dx}[f+g] = \frac{d}{dx}[f] + \frac{d}{dx}[g]$.

Homomorphism Properties: Suppose $\varphi: G \to K$ is homo.

1. $\varphi(e_G) = e_K$ Reason $\varphi(a) = \varphi(a e_G) = \varphi(a)\varphi(e_G)$

   $e_K = \varphi(e_G)$. (cancel.)

2. $\varphi(a^{-1}) = \varphi(a)^{-1}$.   Reason $\varphi(a)\underbrace{\varphi(a^{-1})}_{\uparrow \text{ acts as } \varphi(a)^{-1}} = \varphi(aa^{-1}) = \varphi(e_G) = e_K$

Definition Kernel of a homomorphism $\varphi: G \to K$ is

$\text{Ker}(\varphi) = \{a \in G \mid \varphi(a) = e\}$.

Observation $\text{Ker}(\varphi) \leq G$.

1. $\text{Ker}(\varphi)$ is closed. If $a, b \in \ker(\varphi)$ then $\varphi(a) = e = \varphi(b)$.
   Thus $\varphi(ab) = \varphi(a)\varphi(b) = ee = e$. So $ab \in \text{Ker}(\varphi)$.
2. $\varphi(e) = e$ so $e \in \text{Ker}(\varphi)$.
3. If $a \in \ker(\varphi)$ then $\varphi(a^{-1}) = \varphi(a)^{-1} = e^{-1} = e$. so $a^{-1} \in \ker(\varphi)$.

Ex $\frac{d}{dx}: \mathcal{P} \to \mathcal{P}$ $\ker\left(\frac{d}{dx}\right) = \{a + 0x + 0x^2 + 0x^3 \mid a \in \mathbb{R}\} \cong \mathbb{R}$.

= constant polynomials.

Ex $|\ |: \mathbb{R}^* \to \mathbb{R}^*$ $\ker(|\ |) = \{1, -1\}$ ← subgroup of $\mathbb{R}^*$

	1	-1
1	1	-1
-1	-1	1

Ex $|\ |: \mathbb{C}^* \to \mathbb{C}^*$ $\ker(|\ |) = U$ ← subgroup of $\mathbb{C}^*$.

Ex $\varphi: \mathbb{Z} \to \mathbb{Z}_5$ $\varphi(n) = n \bmod 5$ $\ker(\varphi) = 5\mathbb{Z}$.

Ex $\varphi: G \to K$ $\varphi(a) = e_K$ $\ker(\varphi) = G$.

Ex $\ln: \mathbb{R}^+ \to \mathbb{R}$ $\ker(\varphi) = \{1\}$

Ex $\phi: \mathbb{Z} \to S_8$ when $\varphi(x) = \big((1,4)\cancel{2,6)}(2,5,7)\big)^x$ $\text{Ker}(\varphi) = 6\mathbb{Z}$

In linear algebra, a linear map $T: V \to W$ is determined entirely by what T does to a basis of V.

If $\mathcal{B} = \{v_1, v_2, \cdots v_n\}$ is a basis for V and you know $T(v_i) = w_i$ for $1 \le i \le n$, then $T(x) = T(\sum a_i v_i) = \sum a_i T(v_i) = \sum a_i w_i$

Homomorphisms of finitely generated groups are entirely analogous.

<u>Ex</u> Suppose $\varphi: \mathbb{Z}_4 \to \mathbb{Z}_5 \times \mathbb{Z}_8$ is a homorphism and I tell you that $\varphi(1) = (2,7)$.

Then:

$$
\begin{aligned}
\varphi(0) &= (0,0) \\
\varphi(1) &= (2,7) \\
\varphi(2) &= \varphi(1+1) = \varphi(1) + \varphi(1) = (2,7) + (2,7) = (4,6) \\
\varphi(3) &= \varphi(2+1) = \varphi(2) + \varphi(1) = (4,6) + (2,7) = (1,5) \\
\varphi(4) &= \varphi(3+1) = \varphi(3) + \varphi(1) = (1,5) + (2,7) = (3,4) \\
&\vdots
\end{aligned}
$$

Heres another similarity between this and linear algebra.

Recall that if $T: V \to W$ is linear and $Null(T) = Ker(T) = \{0\}$ then T is 1-1.

<u>Theorem</u> Homomorphism $\varphi: G \to K$ is one-to-one $\iff$ $Ker(\varphi) = \{e\}$.

<u>Proof</u> $\Rightarrow$ Suppose φ is 1-1. ~~If $\varphi(a) = e_K$ Then $\varphi(a) = \varphi(e_K)$, so $a = e_K$~~. If $a \in Ker(\varphi)$, then $\varphi(a) = e_K$ so $\varphi(a) = \varphi(e_G)$. Then $a = e_G$. Conclusion: $Ker(\varphi) = \{e\}$.

$\Leftarrow$ Suppose $Ker(\varphi) = \{e\}$

If $\varphi(a) = \varphi(b)$ then $\varphi(a)\varphi(b)^{-1} = e_K$

$\varphi(a)\varphi(b^{-1}) = e_K$ $\quad \varphi(ab^{-1}) = e_K$.

Thus $ab^{-1} \in Ker(\varphi) = e_G$, so $ab^{-1} = e_G$. So $a = b$.

Example

Find the kernel of the homomorphism $\varphi: \mathbb{Z}_{40} \longrightarrow \mathbb{Z}_5 \times \mathbb{Z}_8$ where $\varphi(1) = (2, 7)$. To do this, first notice that $\varphi(n) = \varphi(1 + 1 + \ldots + 1) = \varphi(1) + \varphi(1) + \ldots + \varphi(1) =$ $(2, 7) + (2, 7) + \ldots + (2, 7) = n(2, 7) = (2n \pmod 5, 7n \pmod 8)$.

Then
$\mathrm{Ker}(\varphi) = \{n \in \mathbb{Z}_{40} \mid \varphi(n) = (0, 0)\} = \{n \in \mathbb{Z}_{40} \mid (2n \pmod 5, 7n \pmod 8) = (0, 0)\}$
$= \{n \in \mathbb{Z}_{40} \mid 5 \text{ divides } 2n \text{ and } 8 \text{ divides } 7n\} = \mathbf{\{0\}}$.

$n \equiv 0, 8, 16, 24, 32$

$\cancel{n} = 0, 5, 10, 15, 20, 25$

Thus the Kernel is trivial, so the homomorphism is one-to-one.

Example

Find the kernel of the homomorphism $\varphi: \mathbb{Z}_{40} \longrightarrow \mathbb{Z}_5 \times \mathbb{Z}_8$ where $\varphi(1) = (0, 2)$. To do this, first notice that $\varphi(n) = \varphi(1 + 1 + \ldots + 1) = \varphi(1) + \varphi(1) + \ldots + \varphi(1) =$ $(0, 2) + (0, 2) + \ldots + (0, 2) = n(0, 2) = (0, 2n \pmod 8)$.

Then, $\mathrm{Ker}(\varphi) = \{n \in \mathbb{Z}_{40} \mid \varphi(n) = (0, 0)\} = \{n \in \mathbb{Z}_{40} \mid (0, 2n \pmod 8) = (0, 0)\} =$ $\{n \in \mathbb{Z}_{40} \mid 8 \text{ divides } 2n\} = \mathbf{\{0, 4, 8, 12, 16, 20, 24, 28, 32, 36\}}$.

Thus the Kernel is NOT trivial, so the homomorphism is NOT one-to-one.

Ex $\quad \varphi: \mathbb{Z}_5 \to \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_6$

$\varphi(x) = (0, 0, x, 0)$

$\mathrm{Ker}(\varphi) = \{0\}$

Ex $\quad \varphi: \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_6 \to \mathbb{Z}_5$

$\varphi(u, v, w, x) = w$

$\mathrm{Ker}(\varphi) = \mathbb{Z}_2 \times \mathbb{Z}_3 \times \{0\} \times \mathbb{Z}_6$

Ex $\quad \varphi: \mathbb{Z} \to S_8 \qquad \varphi(1) = (1, 4, 2, 6)(2, 5, 7) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 5 & 3 & 2 & 7 & 1 & 6 & 8 \end{pmatrix}$

$= (1, 4, 2, 5, 7, 6)$

Proved in Homework:

If $H \leq G$ has property $g^{-1}hg \in H$ $\forall g \in G$, $h \in H$, then $aH = Ha$ $\forall a \in G$.

Note $Ker(\varphi)$ has this property.

Suppose $\varphi : G \to K$ is a homo and $H = Ker(\varphi)$.

Take $h \in H$, $g \in G$

$$\varphi(g^{-1}hg) = \varphi(g^{-1})\varphi(h)\varphi(g) = \varphi(g^{-1})e\varphi(g) = \varphi(gg^{-1}) = \varphi(e) = e.$$

Therefore $g^{-1}hg \in H$.

Theorem: If $\varphi : G \to K$ is a homomorphism and $H = Ker(\varphi)$ Then $aH = Ha$. $\forall a \in G$.

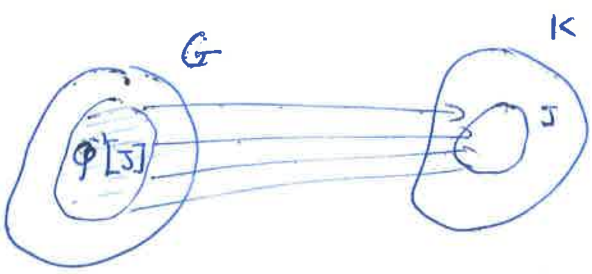
Other definitions and ideas

Suppose $\varphi: G \to K$ is a homomorphism, $H \leq G$, $J \leq K$

Image of G in K: $\varphi[G] = \{\varphi(g) \mid g \in G\} \leq K$

Image of H in K: $\varphi[H] = \{\varphi(h) \mid h \in H\} \leq K$

Inverse image of J: $\varphi^{-1}[J] = \{x \in G \mid \varphi(x) \in J\} \leq G$



Thus, in particular, $Ker(\varphi) = \varphi^{-1}[e]$