Section 11 Direct Products

Suppose G and K are groups.

$G \times K = \{(g,k) \mid g \in G,\ k \in K\}$

Binary operation on $G \times K$: $(g,k) * (g',k') = (gg', kk')$

$(g,k)(g',k') = (gg', kk')$

> Note that set $G \times K$ is actually a group.
>
> $\mathcal{G}_1$ $\big((g,k)(g',k')\big)(g'',k'') = (gg', kk')(g'',k'') = (gg'g'', kk'k'') = (g,k)\big((g',k')(g'',k'')\big)$
>
> $\mathcal{G}_2$ $(e,e) \in G \times K$ is identity: $(e,e)(g,k) = (g,k)$
>
> $\mathcal{G}_3$ $(g^{-1}, k^{-1}) \in G \times K$ is inverse of (g,k). $(g^{-1}k^{-1})(g,k) = (e,e)$

Group $G \times K$ is the <u>direct product</u> of <u>G</u> and <u>K</u>.

<u>Ex</u> $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ $(x,y) + (z,w) = (x+y, z+w)$ ← Vector space group operation is vector addition

<u>Ex</u> $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0,0), (0,1), (1,0), (1,1)\}$

	(0,0)	(0,1)	(1,0)	(1,1)
(00)	(00)	(01)	(10)	(11)
(01)	(01)	(00)	(11)	(10)
(10)	(10)	(11)	(00)	(01)
(11)	(11)	(10)	(01)	(00)

$(1,0)(1\ 1) = (1+1, 0+1) = (0,1)$

Thus $\mathbb{Z}_2 \times \mathbb{Z}_2 \cong V$.

<u>Ex</u> $\mathbb{Z}_2 \times \mathbb{Z}_3 = \{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2)\}$

<u>Note</u> $\langle(1,1)\rangle = \{0(1,1),\ 1(1,1),\ 2(1,1)\ 3(1,1)\ 4(1\,1)\ 5(1,1)\ \}$

$= \{(0,0)\ (1\,1)\ (0,2)\ (1,0)\ (0,1)\ (1\,2)\} = \mathbb{Z}_2 \times \mathbb{Z}_3$

Thus $\mathbb{Z}_2 \times \mathbb{Z}_3$ is <u>cyclic</u>, so $\mathbb{Z}_2 \times \mathbb{Z}_3 \cong \mathbb{Z}_6$.

Ex $\mathbb{Z}_2 \times \mathbb{Z}_4$ This is <u>not</u> cyclic.

Reason: If $(a,b) \in \mathbb{Z}_2 \times \mathbb{Z}_4$ then $4(a,b) = (a,b)+(a,b)+(a,b)+(a,b)$
$= (a+a+a+a,\ b+b+b+b) = (4a, 4b) = (0,0)$.
Thus $|\langle (a,b) \rangle| \le 4$, $|\mathbb{Z}_2 \times \mathbb{Z}_4| = 8$.
So $\mathbb{Z}_2 \times \mathbb{Z}_4 \neq \langle (a,b) \rangle$.

$$\boxed{\text{Theorem} \quad \mathbb{Z}_m \times \mathbb{Z}_n \cong \mathbb{Z}_{mn} \iff m, n \text{ rel prime}}$$

General Situation

If $G_1\ G_2 \cdots G_n$ are groups, Then $G_1 \times G_2 \times \cdots \times G_n = \prod_{i=1}^{n} G_i$
is a group with operation $(a_1, a_2, \cdots, a_n)(b_1, b_2 \cdots b_n) = (a_1b_1, a_2b_2 \cdots a_nb_n)$

Theorem $\prod_{i=1}^{n} \mathbb{Z}_{m_i} \cong \mathbb{Z}_{m_1 m_2 m_3 \cdots m_n} \iff$ (each pair m_i, m_j is rel. prime)

Ex $\mathbb{Z}_{60} \cong \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_5$ since $60 = 4 \cdot 3 \cdot 5$.

$\mathbb{Z}_{60} \ncong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5$

Ex Find order of $(a,b,c) \in \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_5$.

order = smallest n with $n(a,b,c) = (na, nb, nc) = (0,0,0)$

$na = 0 \leftarrow n$ ~~divides order of~~ a
$nb = 0 \leftarrow n$ multiple of order of b
$nc = 0 \leftarrow n$ " " " " c

Thus order is lcm of orders of a, b, c.

Order of $(1,1,1)$ is $\text{lcm}(4,3,5) = 60$
Order of $(2,$ [illegible]$)$ is $\text{lcm}(2,3,5) = 30$
Order of $(3,2,4)$ is $\text{lcm}(4,3,5) = 60$

This illustrates a general Theorem

Theorem If $(a_1\ a_2 \cdots a_n) \in \prod_{i=1}^{n} G_i$ and a_i has order r_i in G_i then $(a_1\ a_2 \cdots a_n)$ has order $\operatorname{lcm}(r_1\ r_2 \cdots r_n)$.

Ex Consider $\left(\begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix}, 2\right) \in S_4 \times \mathbb{Z}_{12}$

order of this element is $\operatorname{lcm}(4, 6) = 12$

Recall G is finitely generated if $G = \langle a, b, c, \ldots, k\rangle$

STRUCTURE OF FINITELY GENERATED ABELIAN GROUPS

Theorem If G is a finitely generated abelian group then $G \cong \mathbb{Z}_{(p_1)^{r_1}} \times \mathbb{Z}_{(p_2)^{r_2}} \times \cdots \times \mathbb{Z}_{(p_n)^{r_n}} \times \mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$ where the p_i's are prime numbers.

Ex Find all finitely generated abelian groups of order 300

$300 = 2^2 \cdot 3 \cdot 5^2$

$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_5$

$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_{25}$ $\quad (\cong V \times \mathbb{Z}_{75})$

$\mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_{25}$ $\quad (\cong \mathbb{Z}_{300})$

$\mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_5$