MATH 504
Algebraic Structures and Functions

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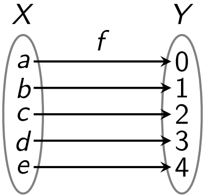
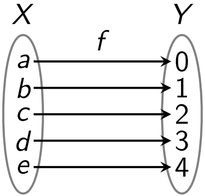
Section 0: Cardinality of Infinite Sets

Recall:

Sets X and Y have the **same cardinality**, written $|X| = |Y|$, provided that there is a one-to-one and onto function $f : X \to Y$.



Recall:
Sets X and Y have the **same cardinality**, written $|X| = |Y|$, provided that there is a one-to-one and onto function $f : X \to Y$.



Theorem $|\mathbb{Z}^+| = |\mathbb{Z}|$.

Proof: Here is a 1-1 onto function $f : X \to Y$.

n	1	2	3	4	5	6	7	8	9	10	11	...
$f(n)$	0	-1	1	-2	2	-3	3	-4	4	-5	5	...

Theorem: $|\mathbb{Q}| = |\mathbb{Z}^+|$.

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Proof: Arrange elements of $\mathbb{Q}$ in the following array.

0	1	-1	2	-2	3	-3	4	-4	5	-5	$\cdots$
$\frac{0}{1}$	$\frac{1}{1}$	$\frac{-1}{1}$	$\frac{2}{1}$	$\frac{-2}{1}$	$\frac{3}{1}$	$\frac{-3}{1}$	$\frac{4}{1}$	$\frac{-4}{1}$	$\frac{5}{1}$	$\frac{-5}{1}$	$\cdots$
	$\frac{1}{2}$	$\frac{-1}{2}$	$\frac{2}{3}$	$\frac{-2}{3}$	$\frac{3}{2}$	$\frac{-3}{2}$	$\frac{4}{3}$	$\frac{-4}{3}$	$\frac{5}{2}$	$\frac{-5}{2}$	$\cdots$
	$\frac{1}{3}$	$\frac{-1}{3}$	$\frac{2}{5}$	$\frac{-2}{5}$	$\frac{3}{4}$	$\frac{-3}{4}$	$\frac{4}{5}$	$\frac{-4}{5}$	$\frac{5}{3}$	$\frac{-5}{3}$	$\cdots$
	$\frac{1}{4}$	$\frac{-1}{4}$	$\frac{2}{7}$	$\frac{-2}{7}$	$\frac{3}{5}$	$\frac{-3}{5}$	$\frac{4}{7}$	$\frac{-4}{7}$	$\frac{5}{4}$	$\frac{-5}{4}$	$\cdots$
	$\frac{1}{5}$	$\frac{-1}{5}$	$\frac{2}{9}$	$\frac{-2}{9}$	$\frac{3}{7}$	$\frac{-3}{7}$	$\frac{4}{9}$	$\frac{-4}{9}$	$\frac{5}{6}$	$\frac{-5}{6}$	$\cdots$
	$\frac{1}{6}$	$\frac{-1}{6}$	$\frac{2}{11}$	$\frac{-2}{11}$	$\frac{3}{8}$	$\frac{-3}{8}$	$\frac{4}{11}$	$\frac{-4}{11}$	$\frac{5}{7}$	$\frac{-5}{7}$	$\cdots$
	$\frac{1}{7}$	$\frac{-1}{7}$	$\frac{2}{13}$	$\frac{-2}{13}$	$\frac{3}{10}$	$\frac{-3}{10}$	$\frac{4}{13}$	$\frac{-4}{13}$	$\frac{5}{8}$	$\frac{-5}{8}$	$\cdots$
	$\frac{1}{8}$	$\frac{-1}{8}$	$\frac{2}{15}$	$\frac{-2}{15}$	$\frac{3}{11}$	$\frac{-3}{11}$	$\frac{4}{15}$	$\frac{-4}{15}$	$\frac{5}{9}$	$\frac{-5}{9}$	$\cdots$
	$\frac{1}{9}$	$\frac{-1}{9}$	$\frac{2}{17}$	$\frac{-2}{17}$	$\frac{3}{13}$	$\frac{-3}{13}$	$\frac{4}{17}$	$\frac{-4}{17}$	$\frac{5}{11}$	$\frac{-5}{11}$	$\cdots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Theorem: $|\mathbb{Q}| = |\mathbb{Z}^+|$.

Proof: Arrange elements of $\mathbb{Q}$ in the following array.

0	1	-1	2	-2	3	-3	4	-4	5	-5	$\cdots$
$\frac{0}{1}$	$\frac{1}{1}$	$\frac{-1}{1}$	$\frac{2}{1}$	$\frac{-2}{1}$	$\frac{3}{1}$	$\frac{-3}{1}$	$\frac{4}{1}$	$\frac{-4}{1}$	$\frac{5}{1}$	$\frac{-5}{1}$	$\cdots$
	$\frac{1}{2}$	$\frac{-1}{2}$	$\frac{2}{3}$	$\frac{-2}{3}$	$\frac{3}{2}$	$\frac{-3}{2}$	$\frac{4}{3}$	$\frac{-4}{3}$	$\frac{5}{2}$	$\frac{-5}{2}$	$\cdots$
	$\frac{1}{3}$	$\frac{-1}{3}$	$\frac{2}{5}$	$\frac{-2}{5}$	$\frac{3}{4}$	$\frac{-3}{4}$	$\frac{4}{5}$	$\frac{-4}{5}$	$\frac{5}{3}$	$\frac{-5}{3}$	$\cdots$
	$\frac{1}{4}$	$\frac{-1}{4}$	$\frac{2}{7}$	$\frac{-2}{7}$	$\frac{3}{5}$	$\frac{-3}{5}$	$\frac{4}{7}$	$\frac{-4}{7}$	$\frac{5}{4}$	$\frac{-5}{4}$	$\cdots$
	$\frac{1}{5}$	$\frac{-1}{5}$	$\frac{2}{9}$	$\frac{-2}{9}$	$\frac{3}{7}$	$\frac{-3}{7}$	$\frac{4}{9}$	$\frac{-4}{9}$	$\frac{5}{6}$	$\frac{-5}{6}$	$\cdots$
	$\frac{1}{6}$	$\frac{-1}{6}$	$\frac{2}{11}$	$\frac{-2}{11}$	$\frac{3}{8}$	$\frac{-3}{8}$	$\frac{4}{11}$	$\frac{-4}{11}$	$\frac{5}{7}$	$\frac{-5}{7}$	$\cdots$
	$\frac{1}{7}$	$\frac{-1}{7}$	$\frac{2}{13}$	$\frac{-2}{13}$	$\frac{3}{10}$	$\frac{-3}{10}$	$\frac{4}{13}$	$\frac{-4}{13}$	$\frac{5}{8}$	$\frac{-5}{8}$	$\cdots$
	$\frac{1}{8}$	$\frac{-1}{8}$	$\frac{2}{15}$	$\frac{-2}{15}$	$\frac{3}{11}$	$\frac{-3}{11}$	$\frac{4}{15}$	$\frac{-4}{15}$	$\frac{5}{9}$	$\frac{-5}{9}$	$\cdots$
	$\frac{1}{9}$	$\frac{-1}{9}$	$\frac{2}{17}$	$\frac{-2}{17}$	$\frac{3}{13}$	$\frac{-3}{13}$	$\frac{4}{17}$	$\frac{-4}{17}$	$\frac{5}{11}$	$\frac{-5}{11}$	$\cdots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Theorem: $|\mathbb{Q}| = |\mathbb{Z}^+|$.

Proof: Arrange elements of $\mathbb{Q}$ in the following array.

0	1	-1	2	-2	3	-3	4	-4	5	-5	$\cdots$
$\frac{0}{1}$	$\frac{1}{1}$	$\frac{-1}{1}$	$\frac{2}{1}$	$\frac{-2}{1}$	$\frac{3}{1}$	$\frac{-3}{1}$	$\frac{4}{1}$	$\frac{-4}{1}$	$\frac{5}{1}$	$\frac{-5}{1}$	$\cdots$
	$\frac{1}{2}$	$\frac{-1}{2}$	$\frac{2}{3}$	$\frac{-2}{3}$	$\frac{3}{2}$	$\frac{-3}{2}$	$\frac{4}{3}$	$\frac{-4}{3}$	$\frac{5}{2}$	$\frac{-5}{2}$	$\cdots$
	$\frac{1}{3}$	$\frac{-1}{3}$	$\frac{2}{5}$	$\frac{-2}{5}$	$\frac{3}{4}$	$\frac{-3}{4}$	$\frac{4}{5}$	$\frac{-4}{5}$	$\frac{5}{3}$	$\frac{-5}{3}$	$\cdots$
	$\frac{1}{4}$	$\frac{-1}{4}$	$\frac{2}{7}$	$\frac{-2}{7}$	$\frac{3}{5}$	$\frac{-3}{5}$	$\frac{4}{7}$	$\frac{-4}{7}$	$\frac{5}{4}$	$\frac{-5}{4}$	$\cdots$
	$\frac{1}{5}$	$\frac{-1}{5}$	$\frac{2}{9}$	$\frac{-2}{9}$	$\frac{3}{7}$	$\frac{-3}{7}$	$\frac{4}{9}$	$\frac{-4}{9}$	$\frac{5}{6}$	$\frac{-5}{6}$	$\cdots$
	$\frac{1}{6}$	$\frac{-1}{6}$	$\frac{2}{11}$	$\frac{-2}{11}$	$\frac{3}{8}$	$\frac{-3}{8}$	$\frac{4}{11}$	$\frac{-4}{11}$	$\frac{5}{7}$	$\frac{-5}{7}$	$\cdots$
	$\frac{1}{7}$	$\frac{-1}{7}$	$\frac{2}{13}$	$\frac{-2}{13}$	$\frac{3}{10}$	$\frac{-3}{10}$	$\frac{4}{13}$	$\frac{-4}{13}$	$\frac{5}{8}$	$\frac{-5}{8}$	$\cdots$
	$\frac{1}{8}$	$\frac{-1}{8}$	$\frac{2}{15}$	$\frac{-2}{15}$	$\frac{3}{11}$	$\frac{-3}{11}$	$\frac{4}{15}$	$\frac{-4}{15}$	$\frac{5}{9}$	$\frac{-5}{9}$	$\cdots$
	$\frac{1}{9}$	$\frac{-1}{9}$	$\frac{2}{17}$	$\frac{-2}{17}$	$\frac{3}{13}$	$\frac{-3}{13}$	$\frac{4}{17}$	$\frac{-4}{17}$	$\frac{5}{11}$	$\frac{-5}{11}$	$\cdots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Get list of all rational numbers.

Theorem: $|\mathbb{Q}| = |\mathbb{Z}^+|$.

Proof: Arrange elements of $\mathbb{Q}$ in the following array.

0	1	−1	2	−2	3	−3	4	−4	5	−5	…
$\frac{0}{1}$	$\frac{1}{1}$	$\frac{-1}{1}$	$\frac{2}{1}$	$\frac{-2}{1}$	$\frac{3}{1}$	$\frac{-3}{1}$	$\frac{4}{1}$	$\frac{-4}{1}$	$\frac{5}{1}$	$\frac{-5}{1}$	…
	$\frac{1}{2}$	$\frac{-1}{2}$	$\frac{2}{3}$	$\frac{-2}{3}$	$\frac{3}{2}$	$\frac{-3}{2}$	$\frac{4}{3}$	$\frac{-4}{3}$	$\frac{5}{2}$	$\frac{-5}{2}$	…
	$\frac{1}{3}$	$\frac{-1}{3}$	$\frac{2}{5}$	$\frac{-2}{5}$	$\frac{3}{4}$	$\frac{-3}{4}$	$\frac{4}{5}$	$\frac{-4}{5}$	$\frac{5}{3}$	$\frac{-5}{3}$	…
	$\frac{1}{4}$	$\frac{-1}{4}$	$\frac{2}{7}$	$\frac{-2}{7}$	$\frac{3}{5}$	$\frac{-3}{5}$	$\frac{4}{7}$	$\frac{-4}{7}$	$\frac{5}{4}$	$\frac{-5}{4}$	…
	$\frac{1}{5}$	$\frac{-1}{5}$	$\frac{2}{9}$	$\frac{-2}{9}$	$\frac{3}{7}$	$\frac{-3}{7}$	$\frac{4}{9}$	$\frac{-4}{9}$	$\frac{5}{6}$	$\frac{-5}{6}$	…
	$\frac{1}{6}$	$\frac{-1}{6}$	$\frac{2}{11}$	$\frac{-2}{11}$	$\frac{3}{8}$	$\frac{-3}{8}$	$\frac{4}{11}$	$\frac{-4}{11}$	$\frac{5}{7}$	$\frac{-5}{7}$	…
	$\frac{1}{7}$	$\frac{-1}{7}$	$\frac{2}{13}$	$\frac{-2}{13}$	$\frac{3}{10}$	$\frac{-3}{10}$	$\frac{4}{13}$	$\frac{-4}{13}$	$\frac{5}{8}$	$\frac{-5}{8}$	…
	$\frac{1}{8}$	$\frac{-1}{8}$	$\frac{2}{15}$	$\frac{-2}{15}$	$\frac{3}{11}$	$\frac{-3}{11}$	$\frac{4}{15}$	$\frac{-4}{15}$	$\frac{5}{9}$	$\frac{-5}{9}$	…
	$\frac{1}{9}$	$\frac{-1}{9}$	$\frac{2}{17}$	$\frac{-2}{17}$	$\frac{3}{13}$	$\frac{-3}{13}$	$\frac{4}{17}$	$\frac{-4}{17}$	$\frac{5}{11}$	$\frac{-5}{11}$	…
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

Get list of all rational numbers. Hence 1-1 onto function $f : \mathbb{Z}^+ \to \mathbb{Q}$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	…
$f(n)$	0	$\frac{1}{1}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{1}$	$\frac{2}{1}$	$\frac{2}{3}$	$\frac{2}{5}$	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{4}$	$-\frac{1}{4}$	$\frac{2}{7}$	…

Thus: $|\mathbb{Z}^+| = |\mathbb{Z}| = |\mathbb{Q}|$.

Notation: $|\mathbb{Z}^+| = \aleph_0$

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But do all infinite sets have cardinality $\aleph_0$?

NO! $|\mathbb{R}| > \aleph_0$

Theorem: $|\mathbb{R}| \neq \aleph_0$, that is, $|\mathbb{R}| \neq |\mathbb{Z}^+|$.

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Proof: Assume to contrary $|\mathbb{R}| = |\mathbb{Z}^+|$. Then there is 1-1 onto $f : \mathbb{Z}^+ \to \mathbb{R}$.

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Proof: Assume to contrary $|\mathbb{R}| = |\mathbb{Z}^+|$. Then there is 1-1 onto $f : \mathbb{Z}^+ \to \mathbb{R}$.

n	$f(n)$
1	0 . 4 0 0 0 0 0 0 0 0 0 0 0 0 0...
2	8 . 5 0 0 6 0 7 0 8 6 6 6 9 0 0...
3	7 . 5 0 5 0 0 9 4 0 0 4 4 1 0 1...
4	5 . 5 0 7 0 4 0 0 8 0 4 8 0 5 0...
5	6 . 9 0 0 2 6 0 0 0 0 0 0 5 0 6...
6	6 . 8 2 8 0 9 5 8 2 0 5 0 0 2 0...
7	6 . 5 0 5 0 5 5 5 0 6 5 5 8 0 8...
8	8 . 7 2 0 8 0 6 4 0 0 0 0 4 4 8...
9	0 . 5 5 0 0 0 0 8 8 8 8 0 0 7 7...
10	0 . 5 0 0 2 0 7 2 2 0 7 8 0 5 1...
11	2 . 9 0 0 0 0 8 8 0 0 0 0 9 0 0...
12	6 . 5 0 2 8 0 0 0 8 0 0 9 6 7 1...
13	8 . 8 9 0 0 8 0 2 4 0 0 8 0 5 0...
14	8 . 5 0 0 0 8 7 4 2 0 8 0 2 2 6...
⋮	⋮

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Proof: Assume to contrary $|\mathbb{R}| = |\mathbb{Z}^+|$. Then there is 1-1 onto $f : \mathbb{Z}^+ \to \mathbb{R}$.

n	$f(n)$
1	0 . 4 0 0 0 0 0 0 0 0 0 0 0 0 0...
2	8 . 5 0 0 6 0 7 0 8 6 6 6 9 0 0...
3	7 . 5 0 5 0 0 9 4 0 0 4 4 1 0 1...
4	5 . 5 0 7 0 4 0 0 8 0 4 8 0 5 0...
5	6 . 9 0 0 2 6 0 0 0 0 0 0 5 0 6...
6	6 . 8 2 8 0 9 5 8 2 0 5 0 0 2 0...
7	6 . 5 0 5 0 5 5 5 0 6 5 5 8 0 8...
8	8 . 7 2 0 8 0 6 4 0 0 0 0 4 4 8...
9	0 . 5 5 0 0 0 0 8 8 8 8 0 0 7 7...
10	0 . 5 0 0 2 0 7 2 2 0 7 8 0 5 1...
11	2 . 9 0 0 0 0 8 8 0 0 0 0 9 0 0...
12	6 . 5 0 2 8 0 0 0 8 0 0 9 6 7 1...
13	8 . 8 9 0 0 8 0 2 4 0 0 8 0 5 0...
14	8 . 5 0 0 0 8 7 4 2 0 8 0 2 2 6...
⋮	⋮

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Proof: Assume to contrary $|\mathbb{R}| = |\mathbb{Z}^+|$. Then there is 1-1 onto $f : \mathbb{Z}^+ \rightarrow \mathbb{R}$.

n	$f(n)$
1	0 . 4 0 0 0 0 0 0 0 0 0 0 0 0 0...
2	8 . 5 0 0 6 0 7 0 8 6 6 6 9 0 0...
3	7 . 5 0 5 0 0 9 4 0 0 4 4 1 0 1...
4	5 . 5 0 7 0 4 0 0 8 0 4 8 0 5 0...
5	6 . 9 0 0 2 6 0 0 0 0 0 0 5 0 6...
6	6 . 8 2 8 0 9 5 8 2 0 5 0 0 2 0...
7	6 . 5 0 5 0 5 5 5 0 6 5 5 8 0 8...
8	8 . 7 2 0 8 0 6 4 0 0 0 0 4 4 8...
9	0 . 5 5 0 0 0 0 8 8 8 8 0 0 7 7...
10	0 . 5 0 0 2 0 7 2 2 0 7 8 0 5 1...
11	2 . 9 0 0 0 0 8 8 0 0 0 0 9 0 0...
12	6 . 5 0 2 8 0 0 0 8 0 0 9 6 7 1...
13	8 . 8 9 0 0 8 0 2 4 0 0 8 0 5 0...
14	8 . 5 0 0 0 8 7 4 2 0 8 0 2 2 6...
⋮	⋮

Let $y =$ 0 . 0 1 0 1 0 0 0 1 0 0 1 0 0 0...

Theorem: $|\mathbb{R}| \neq \aleph_0$, that is, $|\mathbb{R}| \neq |\mathbb{Z}^+|$.

Proof: Assume to contrary $|\mathbb{R}| = |\mathbb{Z}^+|$. Then there is 1-1 onto $f : \mathbb{Z}^+ \to \mathbb{R}$.

n	$f(n)$
1	0 . 4 0 0 0 0 0 0 0 0 0 0 0 0 0...
2	8 . 5 0 0 6 0 7 0 8 6 6 6 9 0 0...
3	7 . 5 0 5 0 0 9 4 0 0 4 4 1 0 1...
4	5 . 5 0 7 0 4 0 0 8 0 4 8 0 5 0...
5	6 . 9 0 0 2 6 0 0 0 0 0 0 5 0 6...
6	6 . 8 2 8 0 9 5 8 2 0 5 0 0 2 0...
7	6 . 5 0 5 0 5 5 5 0 6 5 5 8 0 8...
8	8 . 7 2 0 8 0 6 4 0 0 0 0 4 4 8...
9	0 . 5 5 0 0 0 0 8 8 8 8 0 0 7 7...
10	0 . 5 0 0 2 0 7 2 2 0 7 8 0 5 1...
11	2 . 9 0 0 0 0 8 8 0 0 0 0 9 0 0...
12	6 . 5 0 2 8 0 0 0 8 0 0 9 6 7 1...
13	8 . 8 9 0 0 8 0 2 4 0 0 8 0 5 0...
14	8 . 5 0 0 0 8 7 4 2 0 8 0 2 2 6...
⋮	⋮

Let y = 0 . 0 1 0 1 0 0 0 1 0 0 1 0 0 0...

be number whose nth decimal place differs from nth decimal place of $f(n)$.

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Proof: Assume to contrary $|\mathbb{R}| = |\mathbb{Z}^+|$. Then there is 1-1 onto $f : \mathbb{Z}^+ \to \mathbb{R}$.

n	$f(n)$
1	0 . 4 0 0 0 0 0 0 0 0 0 0 0 0 0...
2	8 . 5 0 0 6 0 7 0 8 6 6 6 9 0 0...
3	7 . 5 0 5 0 0 9 4 0 0 4 4 1 0 1...
4	5 . 5 0 7 0 4 0 0 8 0 4 8 0 5 0...
5	6 . 9 0 0 2 6 0 0 0 0 0 0 5 0 6...
6	6 . 8 2 8 0 9 5 8 2 0 5 0 0 2 0...
7	6 . 5 0 5 0 5 5 5 0 6 5 5 8 0 8...
8	8 . 7 2 0 8 0 6 4 0 0 0 0 4 4 8...
9	0 . 5 5 0 0 0 0 8 8 8 8 0 0 7 7...
10	0 . 5 0 0 2 0 7 2 2 0 7 8 0 5 1...
11	2 . 9 0 0 0 0 8 8 0 0 0 0 9 0 0...
12	6 . 5 0 2 8 0 0 0 8 0 0 9 6 7 1...
13	8 . 8 9 0 0 8 0 2 4 0 0 8 0 5 0...
14	8 . 5 0 0 0 8 7 4 2 0 8 0 2 2 6...
⋮	⋮

Let $y =$ 0 . 0 1 0 1 0 0 0 1 0 0 1 0 0 0...

be number whose nth decimal place differs from nth decimal place of $f(n)$.

Then $y \neq f(n)$ for all $n \in \mathbb{Z}^+$. So f is not onto, a contradiction.