Section 9 Solutions

2. The orbits of $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 2 & 4 & 8 & 3 & 1 & 7 \end{pmatrix}$ are $\{1,5,8,7\}$, $\{2,6,3\}$ and $\{4\}$.

6. The orbits of the permutation $\sigma : \mathbb{Z} \to \mathbb{Z}$ defined as $\sigma(n) = n - 3$ are computed as follows.

$$\begin{array}{cccccccccccccccccccccc} \cdots & 12 & \to & 9 & \to & 6 & \to & 3 & \to & 0 & \to & -3 & \to & -6 & \to & -9 & \to & -12 & \to & \cdots \\ \cdots & 10 & \to & 8 & \to & 5 & \to & 2 & \to & -1 & \to & -4 & \to & -7 & \to & -10 & \to & -13 & \to & \cdots \\ \cdots & 9 & \to & 7 & \to & 4 & \to & 1 & \to & -2 & \to & -5 & \to & -8 & \to & -11 & \to & -14 & \to & \cdots \end{array}$$

Thus, there are three orbits $\{3n | n \in \mathbb{Z}\}$, $\{3n-1 | n \in \mathbb{Z}\}$ and $\{3n-2 | n \in \mathbb{Z}\}$.

8. $(1,3,2,7)(4,8,6) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 7 & 2 & 8 & 5 & 4 & 1 & 6 \end{pmatrix}$

10. $(1,3,2,7)(4,8,6) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 2 & 6 & 3 & 7 & 4 & 5 & 1 \end{pmatrix} = (1,8)(3,6,4)(5,7) = (1,8)(3,6)(6,4)(5,7)$

18. What is the maximum possible order of an element in S_{15}?
Consider an element $\tau \in S_{15}$ that is a product of disjoint cycles of length 7, 5, and 3, respectively. For example, one possibility is $\tau = (1,2,3,4,5,6,7)(8,9,10,11,12)(13,14,15)$.
Then τ has order $7 \cdot 5 \cdot 3 = \mathbf{105}$. A moment of reflection will convince you you can't do any better than this.

29. If H is a subgroup of S_n, then either all the elements of H are even, or exactly half are even and the other half are odd.

Proof. Suppose H is a subgroup of S_n. Notice that H contains the identity permutation, which is even, so H does not consist entirely of odd permutations. If it happens that all the elements of H are even, then there is nothing to prove.

Thus, suppose some elements of H are even and others are odd. Let E be the set of even permutations in H and let O be the set of odd permutations in H. We want to show that E and O have the same cardinality, which means we want to exhibit a one-to-one and onto function $\varphi : E \to O$.

Choose an odd permutation $\sigma \in O$, and define φ by the rule $\varphi(x) = \sigma x$. Notice that this function makes sense. If $x \in E$, then x is even, and since σ is odd, $\varphi(x) = \sigma x$ is odd (odd·even = odd). Moreover, since x and σ are in H, then $\varphi(x) = \sigma x$ is in H as well, because H is closed. Consequently, φ sends even permutations in H to odd permutations in H, that is it is a function from E to O, as advertised.

To see that φ is one-to-one, suppose $\varphi(\tau) = \varphi(\mu)$. This means $\sigma\tau = \sigma\mu$, so $\tau = \mu$ by cancellation on S_n. Thus φ is one-to-one.

To see that φ is onto, let μ be an arbitrary permutation in O. Then $\sigma^{-1}\mu$ is an even permutation (odd·odd = even), and it's in H because both σ (hence σ^{-1}) and μ are in H. Consequently $\sigma^{-1}\mu \in E$. Observe that $\varphi(\sigma^{-1}\mu) = \sigma\sigma^{-1}\mu = \mu$, and it follows that φ is onto.

This completes the demonstration that there is a one-to-one and onto function $\varphi : E \to O$, so $|E| = |O|$. Thus half the permutations of H are even and the other half are odd.