Section 5 Solutions

8. Let H be the set of $n \times n$ matrices whose determinant is 2.

   Note that H is **NOT** a subgroup of $\mathrm{GL}(n,\mathbb{R})$ because it is not closed under matrix multiplication: Suppose $A \in H$. This means $\det(A) = 2$, so $\det(AA) = \det(A)\det(A) = 2 \cdot 2 = 4 \neq 2$, so the product AA does not have determinant 2, so it is not in H. Also H can't be a subgroup because the identity I satisfies $\det(I) = 1 \neq 2$, forcing $I \notin G$. Finally, if $A \in H$ then $\det(A) = 2$ and from linear algebra $\det(A^{-1}) = \frac{1}{2} \neq 2$. In other words $A \in H$ implies $A^{-1} \notin H$.

12. Let H be the set of $n \times n$ matrices whose determinant is 1 or -1.
    Then H is a subgroup of $\mathrm{GL}(n,\mathbb{R})$ for the following reasons.

    (a) First we show H is closed. Suppose $A, B \in H$, which means $\det(A) \in \{1, -1\}$ and $\det(B) \in \{1, -1\}$. Then $\det(AB) = \det(A)\det(B)$ can only be 1 or -1. But this means AB satisfies the requirement for being in H, so $AB \in H$, hence H is closed.

    (b) The identity I is in H because $\det(I) = 1$, meaning I meets the requirement for being in H.

    (c) Suppose $A \in H$. This means $\det(A)$ is either 1 or -1. Hence $\det(A^{-1}) = 1/\det(A)$ is either 1 or -1, so $A^{-1} \in H$.

    Properties 1–3 above show that H is a subgroup of $\mathrm{GL}(n,\mathbb{R})$.

22. Denote the given matrix as A and observe that $A^2 = I$, so A is its own inverse. From this, note that $A^k = A$ if k is odd and $A^k = I$ if k is even. The cyclic subgroup geberated by A is thus $\langle A \rangle = \{A^k \mid k \in \mathbb{Z}\} = \{I, A\}$, and its order is 2. Note that $\langle A \rangle = \{I, A\} \cong \mathbb{Z}_2$.

31. Since $\cos(3\pi/2) + i\sin(3\pi/2) = -i$, the subgroup in question consists of all the integer powers of $-i$. Now, $(-i)^1 = -i$, $(-i)^2 = -1$, $(-i)^3 = i$, $(-i)^4 = 1$. Then $(-i)^5 = -i$ completes the cycle and the pattern continues after this. Thus the subgroup is $\{1, i, -1, -i\}$ and its order is 4.

47. Suppose G is an abelian group. Then $H = \{x \in G | x^2 = e\}$ is a subgroup of G.

    Proof. We make the following observations:

    (a) First we show H is closed. Suppose $x, y \in H$, which means $x^2 = e$ and $y^2 = e$. Using this with the fact that G is abelian we get $(xy)^2 = (xy)(xy) = xyxy = xxyy = x^2y^2 = ee = e$. Now, the fact that $(xy)^2 = e$ means $xy \in H$, so H is closed.

    (b) Observe $e \in H$ because $e^2 = e$ means e satisfies the requirement for being in H.

    (c) Suppose $a \in H$. This means $a^2 = e$, or $aa = e$. Taking inverses of both sides gives $(aa)^{-1} = e^{-1}$, or $a^{-1}a^{-1} = e$, that is, $(a^{-1})^2 = e$, which means $a^{-1} \in H$.

    Properties 1–3 above show that H is a subgroup of G.

51. Suppose G is a group and $a \in G$. Show that $H_a = \{x \in G | xa = ax\}$ is a subgroup of G.

    Proof. We make the following observations:

    (a) First we show H_a is closed. Suppose $x, y \in H_a$, which means $xa = ax$ and $ya = ay$. Using these facts combined with associativity of G, we get $(xy)a = x(ya) = x(ay) = (xa)y = (ax)y = a(xy)$. Thus $(xy)a = a(xy)$, so xy meets the requirement for being in H_a, so $xy \in H_a$. This shows H_a is closed.

    (b) Observe $e \in H_a$ because $ea = ae$, which means e satisfies the requirement for being in H_a.

    (c) Suppose $x \in H_a$. This means $xa = ax$. Left-multiplying both sides by x^{-1} gives $a = x^{-1}ax$. Right-multiplying both sides of *this* by x^{-1} gives $ax^{-1} = x^{-1}a$, which means $x^{-1} \in H_a$.

    Properties 1–3 above show that H is a subgroup of G.