Section 2 Solutions

(2) $(a*b)*c = b*c = a$
$a*(b*c) = a*a = a$

Even though we've shown that $(a*b)*c = a*(b*c)$, that's no guarantee that the opeation $*$ is associative. We would have to show $(x*y)*z = x*(y*z)$ for *all* possible values of x, y and z. In fact, note that $(d*a)*b = b*b = c$ is unequal to $d*(a*b) = d*b = e$, so $*$ is **NOT ASSOCIATIVE.**

(4) The operation $*$ is **NOT COMMUTATIVE** because, for instance, $e*b = b$ but $b*e = c$.

(6) Suppose the following partial table is for an associative binary operation on $S = \{a, b, c, d\}$.

$*$	a	b	c	d
a	a	b	c	d
b	b	a	c	d
c	c	d	c	d
d				

The missing line should give the values of $d*x$ for the various x. To fill in this line, use the fact that the table gives $c*b = d$, together with the fact that $*$ is associative:

$d*a = (c*b)*a = c*(b*a) = c*b = d$
$d*b = (c*b)*b = c*(b*b) = c*a = c$
$d*c = (c*b)*c = c*(b*c) = c*c = c$
$d*d = (c*b)*d = c*(b*d) = c*d = d$

Thus the completed table is as follows

$*$	a	b	c	d
a	a	b	c	d
b	b	a	c	d
c	c	d	c	d
d	d	c	c	d

(10) Consider the binary operation on $\mathbb{Z}$ defined as $a*b = 2^{ab}$.
This is **COMMUTATIVE** because $a*b = 2^{ab} = 2^{ba} = b*a$ for all $a, b \in \mathbb{Z}$.
This is **NOT ASSOCIATIVE** because, in particular
$0*(1*2) = 0*(2^{1\cdot 2}) = 0*4 = 2^{0\cdot 4} = 2^0 = 1$ but
$(0*1)*2 = (2^{0\cdot 1})*2 = 1*2 = 2^{1\cdot 2} = 2^2 = 4$.

(36) Suppose $*$ is an associative binary operation on a set S, and $H = \{a \in S | a*x = x*a \text{ for all } x \in S\}$. Show H is closed under $*$.

Proof. Suppose that a and b are two arbitrary elements of H. To show H is closed, we must show that $a*b \in H$. And to show $a*b$ is in H we must show $a*b$ satisfies the requirement for being in H, that is we must show $(a*b)*x = x*(a*b)$ for every element x in S.

Let x be an arbitrary element of S. The fact that a and b are in H means

$$a*x = x*a \quad (1)$$
$$b*x = x*b \quad (2)$$

Using (1) and (2) together with associativity of $*$, we deduce

$$(a*b)*x = a*(b*x) = a*(x*b) = (a*x)*b = (x*a)*b = x*(a*b).$$

Thus $(a*b)*x = x*(a*b)$, which means $a*b \in H$, so H is closed.