6. $(-3,5)(2,-4) = (-6,-20) = (2-(2\cdot 4), 2-(2\cdot 11)) = (2,2)$ in $\mathbb{Z}_2 \times \mathbb{Z}_{11}$.

12. Consider the set $R = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$. Is R a ring? Is R a field?

    Notice that $R \subset \mathbb{R}$, so we are asking if R is a subgroup of $\mathbb{R}$.

    First, note that R is an additive subgroup of $\mathbb{R}$:

    (a) If $x, y \in R$ then $x = a + b\sqrt{2}$ and $y = a' + b'\sqrt{2}$ for appropriate rational numbers a, b, a', b'. Then $x + y = (a + b\sqrt{2}) + (a' + b'\sqrt{2}) = (a + a') + (b + b')\sqrt{2} \in R$, so R is closed under addition.

    (b) Observe $0 = 0 + 0\sqrt{2} \in R$.

    (c) If $x \in R$ then $x = a + b\sqrt{2}$, so $-x = -a - b\sqrt{2}$ is also in R.

    Next note that R is closed under multiplication, for if $x, y \in R$ then $x = a+b\sqrt{2}$ and $y = a'+b'\sqrt{2}$ for appropriate rational numbers a, b, a', b'. So $xy = (a+b\sqrt{2})(a'+b'\sqrt{2}) = (aa'+2bb')+(ab'+a'b)\sqrt{2} \in R$.

    It follows that R is a subring of $\mathbb{R}$, so $\boxed{R \text{ is a ring.}}$

    Is R a field? Well, R is commutative because it's a subring of $\mathbb{R}$, and R contains the multiplicative identity $1 = 1+0\sqrt{2}$. We just need to show that any nonzero element $a+b\sqrt{2}$ of R has a multiplicative inverse in R. Of course, in $\mathbb{R}$, $(a + b\sqrt{2})^{-1} = \dfrac{1}{a + b\sqrt{2}}$, but the obvious question is if $\dfrac{1}{a + b\sqrt{2}}$ is in R. Observe:

    $$\frac{1}{a+b\sqrt{2}} = \frac{1}{a+b\sqrt{2}}\,\frac{a-b\sqrt{2}}{a-b\sqrt{2}} = \frac{a-b\sqrt{2}}{a^2-2b^2} = \frac{a}{a^2-2b^2} - \frac{b}{a^2-2b^2}\sqrt{2}$$

    Since $a, b \in \mathbb{Q}$, it follows $\dfrac{a}{a^2-2b^2}$ and $-\dfrac{b}{a^2-2b^2}$ are in $\mathbb{Q}$ too, hence $\dfrac{1}{a+b\sqrt{2}} \in R$. $\boxed{\text{Thus } R \text{ is a field.}}$

18. Find all units in $\mathbb{Z} \times \mathbb{Q} \times \mathbb{Z}$.

    Suppose (a, b, c) is a unit in $\mathbb{Z} \times \mathbb{Q} \times \mathbb{Z}$.
    This means there is an element $(a', b', c') \in \mathbb{Z}\times\mathbb{Q}\times\mathbb{Z}$ with $(a, b, c)(a', b', c') = (aa', bb', cc') = (1, 1, 1)$.
    Since $a, a' \in \mathbb{Z}$ and $aa' = 1$ it follows that $a = \pm 1$.
    Since $b, b' \in \mathbb{Z}$ and $bb' = 1$ it follows that $b = \pm 1$.
    Since $c, c' \in \mathbb{Q}$ and $cc' = 1$ it follows that $c \neq 0$ (and $c' = 1/c$).
    Thus the units in $\mathbb{Z} \times \mathbb{Q} \times \mathbb{Z}$ are all the elements of form $(\pm 1, c, \pm 1)$ with $c \neq 0$
    That is, the units are $\{1, -1\} \times \mathbb{Q}^* \times \{1, -1\}$.

50. Suppose a is an element of a ring R and $I_a = \{x \in R | ax = 0\}$. Show I_a is a subring of R.

    Proof. First note that I_a is an additive subgroup of R:

    (a) I_a is closed under addition: If $x, y \in I_a$ then $ax = 0$ and $yx = 0$. Then $0 = ax + ay = a(x + y)$. But $a(x + y) = 0$ means $x + y \in I_n$.

    (b) The additive identity 0 is in I_a because $a0 = 0$.

    (c) If $x \in I_n$, then $ax = 0$, hence $a(-x) = -(ax) = -0 = 0$, meaning $-x \in I_a$.

    Now we just need to check I_a is closed under multiplication.
    Suppose $x, y \in I_a$ so $ax = 0$ and $yx = 0$.
    Then $a(xy) = (ax)y = 0y = 0$.
    But $a(xy) = 0$ means $xy \in I_a$, so I_a is closed under multiplication.

    Therefore I_a is a subring of R.