Section 14 Solutions

10. Find the order of the element $26 + \langle 12 \rangle \in \mathbb{Z}_{60}/\langle 12 \rangle$.

    The order of this element is the smallest positive integer k for which $k(26 + \langle 12 \rangle)$ equals the identity in $\mathbb{Z}_{60}/\langle 12 \rangle$. That is, it is the smallest positive integer k for which $k(26 + \langle 12 \rangle) = 0 + \langle 12 \rangle = \langle 12 \rangle$.

    Note that $\langle 12 \rangle = \{0, 12, 24, 36, 48\}$.

    Observe $1(26 + \langle 12 \rangle) = 26 + \langle 12 \rangle \neq \langle 12 \rangle$
    $2(26 + \langle 12 \rangle) = 52 + \langle 12 \rangle \neq \langle 12 \rangle$
    $3(26 + \langle 12 \rangle) = 18 + \langle 12 \rangle \neq \langle 12 \rangle$
    $4(26 + \langle 12 \rangle) = 44 + \langle 12 \rangle \neq \langle 12 \rangle$
    $5(26 + \langle 12 \rangle) = 10 + \langle 12 \rangle \neq \langle 12 \rangle$
    $6(26 + \langle 12 \rangle) = 0 + \langle 12 = \rangle \langle 12 \rangle$

    Therefore the order is 6.

14. Find the order of the element $(3,3) + \langle (1,2) \rangle$ in $(\mathbb{Z}_4 \times \mathbb{Z}_8)/\langle (1,2) \rangle$.

    The order of this element is the smallest positive integer k for which $k((3,3) + \langle (1,2) \rangle)$ equals the identity in $(\mathbb{Z}_4 \times \mathbb{Z}_8)/\langle (1,2) \rangle$. That is, it is the smallest positive integer k for which $k((3,3) + \langle (1,2) \rangle) = (0,0) + \langle (1,2) \rangle = \langle (1,2) \rangle$.

    Note that $\langle (1,2) \rangle = \{(0,0), (1,2), (2,4), (3,6)\}$.

    Observe $1((3,3) + \langle (1,2) \rangle) = (3,3) + \langle (1,2) \rangle \neq \langle (1,2) \rangle$
    $2((3,3) + \langle (1,2) \rangle) = (2,6) + \langle (1,2) \rangle \neq \langle (1,2) \rangle$
    $3((3,3) + \langle (1,2) \rangle) = (1,1) + \langle (1,2) \rangle \neq \langle (1,2) \rangle$
    $4((3,3) + \langle (1,2) \rangle) = (0,4) + \langle (1,2) \rangle \neq \langle (1,2) \rangle$
    $5((3,3) + \langle (1,2) \rangle) = (3,7) + \langle (1,2) \rangle \neq \langle (1,2) \rangle$
    $6((3,3) + \langle (1,2) \rangle) = (2,2) + \langle (1,2) \rangle \neq \langle (1,2) \rangle$
    $7((3,3) + \langle (1,2) \rangle) = (1,5) + \langle (1,2) \rangle \neq \langle (1,2) \rangle$
    $8((3,3) + \langle (1,2) \rangle) = (0,0) + \langle (1,2) \rangle \neq \langle (1,2) \rangle$

    Therefore the order is 8.

24. Show that A_n is a normal subgroup of S_n and find a known group that S_n/A_n is isomorphic to.

    Recall that A_n is the set of even permutations in S_n. Given any permutation $\pi \in S_n$, and some even permutation $\alpha \in A_n$ note that $\pi\alpha\pi^{-1}$ is still an even permutation because depending on whether π is even or odd, $\pi\alpha\pi^{-1}$ is either a product of three even permutations or a product of two odds and one even, so either way it is even. It follows that $\pi A_n \pi^{-1} = A_n$, which means A_n is normal.

    Since S_n has twice as many elements as A_n it follows that S_n/A_n has just two elements, so it is isomorphic to $\mathbb{Z}_2$.

34. Show that if G has exactly one subgroup H of a particular order, then H is normal.

Proof. Suppose G has exactly one subgroup H of a particular order n. Take any element $g \in G$ and consider the set $gHg^{-1} = \{ghg^{-1} \mid h \in H\}$. We claim that gHg^{-1} is a subgroup of G:

- Note that $e \in gHg^{-1}$ because $e = ghg^{-1}$ for $h = e \in H$.
- Take two elements ghg^{-1} and $gh'g^{-1}$ in gHg^{-1}. Their product is $ghg^{-1}gh'g^{-1} = ghh'g^{-1}$, which is in gHg^{-1} because $hh' \in H$ by closure of H.
- Take any element $ghg^{-1} \in gHg^{-1}$. Its inverse is $(ghg^{-1})^{-1} = (g^{-1})^{-1}h^{-1}g^{-1} = gh^{-1}g^{-1}$, and this in in gHg^{-1} because $h^{-1} \in H$.

The above shows that gHg^{-1} is a subgroup of G.

We now claim that subgroup gHg^{-1} has the same order as H. Indeed, form the map $\varphi : H \to gHg^{-1}$ as $\varphi(h) = ghg^{-1}$. Note that φ is injective because if $\varphi(x) = \varphi(y)$, then $gxg^{-1} = gyg^{-1}$, so $x = y$ by cancellation.

Also φ is surjective because given any element $ghg^{-1} \in gHg^{-1}$ we have $\varphi(h) = ghg^{-1}$.

Thus gHg^{-1} is a subgroup of G and it has the same order as H. But we know H is the only subgroup of that particular order, so it must be that $gHg^{-1} = H$. This means H is normal.

35. Show that if H and N are subgroups of G and N is normal, then the $H \cap N$ is normal in H.

Proof. Suppose H and N are subgroups of G and N is normal. We need to show $H \cap N$ is normal in H. To do this it suffices to show that for any $h \in H$ and $y \in H \cap N$, it is the case that $hyh^{-1} \in H \cap N$. Let h and y be as stated. Now, $y \in H \cap N$, so $y \in N$. And since N is normal we have $hyh^{-1} \in N$. But also $y \in H$, so $hyh{-}1 \in H$. Thus we have $hyh^{-1} \in H \cap N$. It follows that N is normal.

To see that $H \cap N$ need not be normal in G, let $G = N = S_3$ and $H = \{\iota, \mu_1\}$. Then N is clearly normal in S_3 but $H \cap N = H$ is not normal, as we have seen in class.