**Section 13 Solutions**

2. Consider $\varphi : \mathbb{R} \to \mathbb{Z}$ be defined as $\varphi(x) =$ the greatest integer $\leq x$.
This is NOT a homomorphism, for the homomorphism property doesn't hold:
$\varphi(1.6 + 1.4) = \varphi(3) = 3 \neq 2 = 1 + 1 = \varphi(1.6) + \varphi(1.4)$

10. Suppose $F$ is the additive group of all functions $\mathbb{R} \to \mathbb{R}$.
Consider $\varphi : F \to \mathbb{R}$ defined as $\varphi(f) = \int_0^4 f(x)dx$. This is a homomorphism because
$\varphi(f + g) = \int_0^4 (f + g)(x)dx = \int_0^4 (f(x) + g(x))dx = \int_0^4 f(x)dx + \int_0^g (x)dx = \varphi(f) + \varphi(g)$.

16. Consider the homomorphism $\varphi : S_3 \to \mathbb{Z}_2$ where $\varphi(\sigma) = \begin{cases} 0 & \text{if } \sigma \text{ is even} \\ 1 & \text{if } \sigma \text{ is odd} \end{cases}$
$\ker(\varphi) = \{\sigma \in S_3 | \text{ if } \sigma \text{ is even}\} = \boxed{A_3 = \{\rho_0, \rho_1, \rho_2\}.}$

18. Consider the homomorphism $\varphi : \mathbb{Z} \to \mathbb{Z}_{10}$ with $\varphi(1) = 6$.

Notice the homomorphism property gives a formula for $\varphi$:
$\varphi(n) = \varphi(1 + 1 + 1 + \cdots + 1) = \varphi(1) + \varphi(1) + \varphi(1) + \cdots \varphi(1) = n\varphi(1) = 6n$.

Thus $\varphi(18) = 6 \cdot 18 = 108 = \boxed{8}$

$\ker(\varphi) = \{n \in \mathbb{Z} | \varphi(m) = 0\} = \{n \in \mathbb{Z} | 6n = 0\} = \boxed{5\mathbb{Z}}$

(Because if $6n = 0$ in $\mathbb{Z}_{10}$, then $6n$ must be a multiple of 10, hence $n$ is a multiple of 5)

24. Let $\varphi : \mathbb{Z} \times \mathbb{Z} \to S_{10}$ be such that $\varphi(1, 0) = (3, 5)(2, 4)$ and $\varphi(0, 1) = (1, 7)(6, 10, 8, 9)$.

As in the previous problem, there is a formula for $\varphi$:

$\varphi((m, n)) = \varphi((m, 0) + (0, n)) = \varphi(m(1, 0))\varphi(n(0, 1)) =$

$[\varphi((1, 0) + (1, 0) + \cdots + (10))][\varphi((0, 1) + (0, 1) + \cdots + (0, 1))] =$

$\varphi(1, 0)^m\varphi(0, 1)^n = [(3, 5)(2, 4)]^m[(1, 7)(6, 10, 8, 9)]^n = (3, 5)^m(2, 4)^m(1, 7)^n(6, 10, 8, 9)^n$

Thus $\varphi(3, 10) = (3, 5)^3(2, 4)^3(1, 7)^{10}(6, 10, 8, 9)^{10} = \boxed{(3, 5)(2, 4)(6, 8)(10, 9)}$

To find the kernel of $\varphi$, note that $\varphi((m, n)) = (3, 5)^m(2, 4)^m(1, 7)^n(6, 10, 8, 9)^n$ will only be the identity permutation if $m$ is a multiple of 2 and $n$ is a multiple of 4. Thus $\boxed{\ker(\varphi) = 2\mathbb{Z} \times 4\mathbb{Z}.}$

50. Let $\varphi : G \to H$ be a homomorphism. Show that $\varphi[G]$ is abelian if and only if for all $x, y \in G$, we have $xyx^{-1}y^{-1} \in \ker(\varphi)$.

Proof. First, suppose $\varphi[G]$ is abelian. Denote the identity in $H$ as $e'$. Recall $\varphi[G] = \{\varphi(x) | x \in G\}$, so $\varphi[G]$ being abelian means that the following equation holds for all $x, y \in G$:

$$\varphi(x)\varphi(y) = \varphi(y)\varphi(x). \tag{1}$$

Now let's check that $xyx^{-1}y^{-1} \in \ker(\varphi)$. Using the homomorphism property for $\varphi$ followed by equation (1), we get $\varphi(xyx^{-1}y^{-1}) = \varphi(x)\varphi(y)\varphi(x^{-1})\varphi(y^{-1}) = \varphi(x)\varphi(x^{-1})\varphi(y)\varphi(y^{-1}) = \varphi(xx^{-1})\varphi(yy^{-1})$ $= \varphi(e)\varphi(e) = e'e' = e'$. This shows $\varphi(xyx^{-1}y^{-1}) = e'$, whence $xyx^{-1}y^{-1} \in \ker(\varphi)$.

Conversely, suppose $xyx^{-1}y^{-1} \in \ker(\varphi)$ for all $x, y \in G$. Take two arbitrary elements $\varphi(x)$ and $\varphi(y)$ in $\varphi[G]$. Since $xyx^{-1}y^{-1} \in \ker(\varphi)$, it follows $\varphi(xyx^{-1}y^{-1}) = e'$, so $\varphi(x)\varphi(y)\varphi(x^{-1})\varphi(y^{-1}) = e'$, which becomes $\varphi(x)\varphi(y)\varphi(x)^{-1}\varphi(y)^{-1} = e'$. Right-multiplying by $\varphi(y)$ and again by $\varphi(x)$ produces $\varphi(x)\varphi(y) = \varphi(y)\varphi(x)$, so $\varphi[G]$ is abelian.