Section 10 Solutions

2. Find the cosets of the subgroup $4\mathbb{Z}$ of $2\mathbb{Z}$.

   There are two cosets:
   $$\begin{aligned} 4\mathbb{Z} &= \{\ldots,-8,-4,0,4,8,12,16,\ldots\} = \{4k \mid k\in\mathbb{Z}\} \\ 2+4\mathbb{Z} &= \{\ldots,-6,-2,2,6,10,14,18,\ldots\} = \{2+4k \mid k\in\mathbb{Z}\} \end{aligned}$$

4. Find all cosets of the subgroup $\langle 4\rangle = \{0,4,8\}$ of $\mathbb{Z}_{12}$.
   $$\begin{aligned} \langle 4\rangle &= \{0,4,8\} \\ 1+\langle 4\rangle &= \{1,5,9\} \\ 2+\langle 4\rangle &= \{2,6,10\} \\ 3+\langle 4\rangle &= \{3,7,11\} \end{aligned}$$

12. Find the index of the subgroup $\langle 3\rangle = \{0,3,6,9,12,15,18,21\}$ in $\mathbb{Z}_{24}$.
    The index is $\dfrac{|\mathbb{Z}_{24}|}{|\langle 3\rangle|} = \dfrac{24}{8} = 3$.

16. Consider $\mu = (1,2,4,5)(3,6) \in S_6$. Find the index of the subgroup $\langle\mu\rangle$.

    Note $\langle\mu\rangle = \{(1,2,4,5)(3,6),\ (1,4)(2,5),\ (1,5,2,1)(3,6),\ ()\}$
    The index is $\dfrac{|S_6|}{|\langle\mu\rangle|} = \dfrac{6!}{4} = 180$.

29. Suppose $H \leq G$ has the property that $g^{-1}hg \in H$ for all $h \in H$ and $g \in G$.
    Show that $gH = Hg$ for every element $g \in G$.

    Proof. Assume g is an arbitrary but fixed element of G.

    First we show $gH \subseteq Hg$. Suppose $x \in gH$. By definition of gH, this means $x = gh$ for some $h \in H$. Then $x = gh = (g^{-1})^{-1}h = (g^{-1})^{-1}hg^{-1}g = [(g^{-1})^{-1}hg^{-1}]g$. The property that H is assumed to have gives $(g^{-1})^{-1}hg^{-1} \in H$, so $x = h'g$ for $h' = (g^{-1})^{-1}hg^{-1} \in H$. Consequently $x \in Hg$, so $gH \subseteq Hg$.

    Next we show $Hg \subseteq gH$. Suppose $x \in Hg$. By definition of Hg, this means $x = hg$ for some $h \in H$. Then $x = hg = g(g^{-1}hg)$. By assumption, $g^{-1}hg \in H$, so $x = gh'$ for $h' = g^{-1}hg \in H$. This means $x \in gH$, so $Hg \subseteq gH$.

    Since $gH \subseteq Hg$ and $Hg \subseteq gH$, it follows that $gH = Hg$