Section 0 Solutions

Use this solution key as a guide in resolving the problems (if any) you had on your homework. It also gives an indication of the level of completeness and detail that I'll be looking for in your homework this semester.

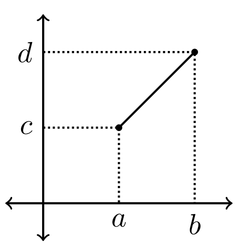
(2) $\{m \in \mathbb{Z} | m^2 = 3\} = \emptyset$ (Empty set because $\sqrt{3}$ and $-\sqrt{3}$ are not in $\mathbb{Z}$.)

(12) (a) Function; not one-to-one because $f(1) = f(2)$; not onto because 2 is not in the range.
(b) Function; not one-to-one because $f(1) = f(3)$; not onto because 2 is not in the range.
(c) Not a function;
(d) Function; both one-to-one and onto.
(e) Function; not one-to-one because $f(1) = f(2)$; not onto because 2,4 not in the range.
(f) Not a function;

(14) (a) Consider the function $f : [0, 1] \to [0, 2]$ defined as $f(x) = 2x$.
Then f is one-to-one, for if $f(a) = f(b)$, then $2a = 2b$, hence $a = b$.
And f is onto, because if $y \in [0, 2]$, then $f(y/2) = y$.
Thus $[0, 1]$ and $[0, 2]$ have the same cardinality.

(b) Consider the linear function $f : [1, 3] \to [5, 25]$ defined as $f(x) = 10x - 5$.
Then f is one-to-one, for if $f(a) = f(b)$, then $10a - 5 = 10b - 5$, hence $10a = 10b$, so $a = b$.
And f is onto, because if $y \in [5, 25]$, then $f((y + 5)/10) = y$.
Thus $[1, 3]$ and $[5, 25]$ have the same cardinality.

(c) To find a one-to-one and onto function $f : [a, b] \to [c, d]$, note that it suffices to use a linear function $f(x) = mx + b_0$ for which $f(a) = c$ and $f(b) = d$, as illustrated below.



The slope of this line is $m = \frac{d-c}{b-a}$. By standard methods from high school algebra the line is the graph of the function

$$f(x) = \frac{d-c}{b-a}x - \frac{d-c}{b-a}a + c.$$

As above, this is a one-to-one onto function $f : [a, b] \to [c, d]$, so the intervals $[a, b]$ and $[c, d]$ have the same cardinality.

(16) (a) $\mathscr{P}(\emptyset) = \{\emptyset\}$ has cardinality 1.
(b) $\mathscr{P}(\{a\}) = \{\emptyset, \{a\}\}$ has cardinality 2.
(c) $\mathscr{P}(\{a, b\}) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ has cardinality 4.
(d) $\mathscr{P}(\{a, b, c\}) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$ has cardinality 8.

(Continued on next page.)

(18) Suppose A is a set, $B = \{0, 1\}$ and B^A is the set of all functions from A to B. Show that B^A and $\mathscr{P}(A)$ have the same cardinality.

Proof: To show that B^A and $\mathscr{P}(A)$ have the same cardinality, we must construct a one-to-one and onto function $\varphi : B^A \to \mathscr{P}(A)$. Define φ by the rule $\varphi(f) = \{x \in A | f(x) = 1\}$. Notice that φ is defined in such a way that any $f \in B^A$ maps to a subset of A (i.e. to an element of $\mathscr{P}(A)$), so φ is a well-defined function mapping B^A to $\mathscr{P}(A)$.

To complete the proof, we must show that φ is both one-to-one and onto.

To show that φ is one-to-one, consider $f, g \in B^A$ with $f \neq g$. This means there is some element $x \in A$ with $f(x) \neq g(x)$. That is, either $f(x) = 1$ and $g(x) = 0$, or $f(x) = 0$ and $g(x) = 1$. Consider these two cases separately. If $f(x) = 1$ and $g(x) = 0$, then, by definition of φ, we have $x \in \varphi(f)$ and $x \notin \varphi(g)$, so $\varphi(f)$ and $\varphi(g)$ are different subsets of A, hence $\varphi(f) \neq \varphi(g)$. By the same reasoning, if $f(x) = 0$ and $g(x) = 1$, then $x \notin \varphi(f)$ and $x \in \varphi(g)$, so again $\varphi(f) \neq \varphi(g)$. Either way, we have shown that $f \neq g$ implies $\varphi(f) \neq \varphi(g)$, which means φ is one-to-one.

To show that φ is onto, consider and arbitrary $X \in \mathscr{P}(A)$, that is, an arbitrary subset $X \subseteq A$. Let $f \in B^A$ be the function defined as

$$f(x) = \begin{cases} 1 & \text{if } x \in X \\ 0 & \text{if } x \notin X \end{cases}$$

Then the definitions of φ and f give $\varphi(f) = \{x \in A | f(x) = 1\} = \{x \in A | x \in X\} = X$. Thus φ is onto.

Thus, since $\varphi : B^A \to \mathscr{P}(A)$ is one-to-one and onto, it follows that B^A and $\mathscr{P}(A)$ have the same cardinality.

(26) Let the set be $\{a, b, c, d\}$ It has 15 different partitions, listed below.

$\{\{a\},\{b\},\{c\},\{d\}\}$	$\{\{a\},\{b\},\{c,d\}\}$	$\{\{a\},\{b,c,d\}\}$	$\{\{a,b\},\{c,d\}\}$	$\{a,b,c,d\}$
	$\{\{a\},\{c\},\{b,d\}\}$	$\{\{b\},\{a,c,d\}\}$	$\{\{a,c\},\{b,d\}\}$	
	$\{\{a\},\{d\},\{b,c\}\}$	$\{\{c\},\{a,b,d\}\}$	$\{\{a,d\},\{b,c\}\}$	
	$\{\{b\},\{c\},\{a,d\}\}$	$\{\{d\},\{a,b,c\}\}$		
	$\{\{b\},\{d\},\{a,c\}\}$			
	$\{\{c\},\{d\},\{a,b\}\}$			

(30) Consider the relation $\mathscr{R}$ on $\mathbb{R}$ defined as $x\mathscr{R}y$ if $x \geq y$. This is **not** an equivalence relation because the symmetric property fails: Observe that $5\mathscr{R}3$, while it is not true that $3\mathscr{R}5$.

(34) Consider the relation $\mathscr{R}$ on $\mathbb{Z}^+$ defined as $m\mathscr{R}n$ if m and n have the same last digit in base-ten notation.

This relation is **reflexive**, for any integer m has the same last digit as itself, so $m\mathscr{R}m$ for every $m \in \mathbb{Z}^+$.

This relation is **symmetric**, for if m and n have the same last digit, then certainly n and m have the same last digit, so $m\mathscr{R}n$ implies $n\mathscr{R}m$ for all $m, n \in \mathbb{Z}^+$.

This relation is **transitive**, for if m and n have the same last digit, and n and p have the same last digit, then certainly m and p have the same last digit, so $m\mathscr{R}n$ and $n\mathscr{R}p$ implies $m\mathscr{R}p$ for all $m, n, p \in \mathbb{Z}^+$.

Thus $\mathscr{R}$ is an equivalence relation. This relation gives rise to 10 equivalence classes, which form a partition of $\mathbb{Z}^+$, listed below.

$\bar{1} = \{1, 11, 21, 31, 41, \ldots\}$ $\quad$ $\bar{2} = \{2, 12, 22, 32, 42, \ldots\}$
$\bar{3} = \{3, 13, 23, 33, 43, \ldots\}$ $\quad$ $\bar{4} = \{4, 14, 24, 34, 44, \ldots\}$
$\bar{5} = \{5, 15, 25, 35, 45, \ldots\}$ $\quad$ $\bar{6} = \{6, 16, 26, 36, 46, \ldots\}$
$\bar{7} = \{7, 17, 27, 37, 47, \ldots\}$ $\quad$ $\bar{8} = \{8, 18, 28, 38, 48, \ldots\}$
$\bar{9} = \{9, 19, 29, 39, 49, \ldots\}$ $\quad$ $\bar{10} = \{\ \ 10, 20, 30, 40, \ldots\}$