Section 3 Solutions

8. Consider the binary structures $\langle M_2(\mathbb{R}), \cdot\rangle$ and $\langle \mathbb{R}, \cdot\rangle$, and the map $\varphi : M_2(\mathbb{R}) \to \mathbb{R}$ defined as $\varphi(A) = \det(A)$. Is φ an isomorphism?

   Notice that a property of determinants gives $\varphi(A \cdot B) = \det(A \cdot B) = \det(A) \cdot \det(B) = \varphi(A) \cdot \varphi(B)$, so φ *does* satisfy the homomorphism property. Also, φ is onto, for if $y \in \mathbb{R}$, then $\varphi\left(\begin{bmatrix} y & 0 \\ 0 & 1 \end{bmatrix}\right) = \det\left(\begin{bmatrix} y & 0 \\ 0 & 1 \end{bmatrix}\right) = y$. So far so good. However, φ is *not* one-to-one because $\varphi\left(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\right) = \varphi\left(\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\right) = 2$, but $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \neq \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$. Therefore φ **is NOT an isomorphism.**

18. (a) Consider the one-to-one and onto map $\varphi : \mathbb{Q} \to \mathbb{Q}$ defined as $\varphi(x) = 3x - 1$. Describe a binary operation $*$ on $\mathbb{Q}$ so that φ is an isomorphism from $\langle \mathbb{Q}, +\rangle$ to $\langle \mathbb{Q}, *\rangle$.

    Note that for any $x \in \mathbb{Q}$ we have $\varphi\left(\frac{x+1}{3}\right) = x$.

    Now we want to find out what $a * b$ equals. From the above line, and from the fact that the condition $\varphi(x) * \varphi(y) = \varphi(x + y)$ must hold, we get:

$$a * b = \varphi\left(\frac{a+1}{3}\right) * \varphi\left(\frac{b+1}{3}\right) = \varphi\left(\frac{a+1}{3} + \frac{b+1}{3}\right) = \varphi\left(\frac{a+b+2}{3}\right) = 3\frac{a+b+2}{3} - 1 = a + b + 1$$

    Therefore our binary operation is $\boxed{a * b = a + b + 1.}$

    For this particular binary operation the element $-1 \in \mathbb{Q}$ is the identity because $-1 * a = -1 + a + 1 = a$.

    (b) Consider the one-to-one and onto map $\varphi : \mathbb{Q} \to \mathbb{Q}$ defined as $\varphi(x) = 3x - 1$. Describe a binary operation $*$ on $\mathbb{Q}$ so that φ is an isomorphism from $\langle \mathbb{Q}, *\rangle$ to $\langle \mathbb{Q}, +\rangle$.

    Since φ must have the homomorphism property, we have

$$\begin{aligned} \varphi(a * b) &= \varphi(a) + \varphi(b) \\ 3(a * b) - 1 &= 3a - 1 + 3b - 1 \\ 3(a * b) &= 3a + 3b - 1 \\ a * b &= a + b - \tfrac{1}{3} \end{aligned}$$

    Thus $*$ is defined as $\boxed{a * b = a + b - \dfrac{1}{3}.}$

    To see that φ is an isomorphism, notice that it satisfies the homomorphism property:

$$\varphi(a * b) = \varphi\left(a + b - \frac{1}{3}\right) = 3\left(a + b - \frac{1}{3}\right) - 1 = 3a + 3b - 2 = (3a - 1) + (3b - 1) = \varphi(a) + \varphi(b).$$

    Since $a * \frac{1}{3} = a = \frac{1}{3} * a$, for all $a \in \mathbb{Q}$, it follows that $\frac{1}{3}$ is the identity.

26. Prove that if $\varphi : S \to S'$ is an isomorphism from $\langle S, *\rangle$ to $\langle S', *'\rangle$, then $\varphi^{-1} : S' \to S$ is an isomorphism from $\langle S', *'\rangle$ to $\langle S, *\rangle$.

    First, since φ is one-to-one and onto, its inverse φ^{-1} is also one-to-one and onto. (One-to-one because if $\varphi^{-1}(a) = \varphi^{-1}(b)$, then $\varphi(\varphi^{-1}(a)) = \varphi(\varphi^{-1}(b))$, so $a = b$; Onto because if $y \in S$, then $\varphi^{-1}(\varphi(y)) = y$.)

    Therefore, we just need to show that φ satisfies the homomorphism property. Given arbitrary elements $x, y \in S'$, notice that

$$\begin{aligned} \varphi^{-1}(x *' y) &= \varphi^{-1}\left[\varphi\big(\varphi^{-1}(x)\big) *' \varphi\big(\varphi^{-1}(y)\big)\right] && \text{(because } x = \varphi(\varphi^{-1}(x)), etc\text{)} \\ &= \varphi^{-1}\left[\varphi\big(\,\varphi^{-1}(x) * \varphi^{-1}(y)\,\big)\right] && \text{(because } \varphi(z) *' \varphi(w) = \varphi(z * w)\text{)} \\ &= \varphi^{-1}(x) * \varphi^{-1}(y) && \text{(because } \varphi^{-1}(\varphi(z)) = z\text{)} \end{aligned}$$

    Thus we have shown that $\varphi^{-1}(x *' y) = \varphi^{-1}(x) * \varphi^{-1}(y)$, which shows that φ^{-1} has the homomorphism property.

    In summary, since $\varphi^{-1} : S' \to S$ is one-to-one and onto and satisfies the homomorphism property, it is an isomorphism of $\langle S', *'\rangle$ with $\langle S, *\rangle$.