VCU

MATH 307
Multivariate Calculus

R. Hammack

Test 3

November 8, 2013

Name: Richard

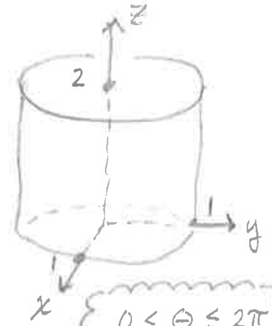
Score: 100

Directions. Solve the following questions in the space provided. Unless noted otherwise, you must show your work to receive full credit. This is a closed-book, closed-notes test. Calculators, computers, etc., are not used. Put a your final answer in a $\boxed{\text{box,}}$ where appropriate.

4. (25 pts.) Suppose D is the cylinder whose base is the unit circle on the xy-plane, and whose top lies on the plane z = 2.

Compute the integral $\iiint_D r^2z^3\,dV$.

(Use cylindrical coordinates.)



$0 \le \theta \le 2\pi$
$0 \le r \le 1$
$0 \le z \le 2$

$$\iiint_D r^2z^3\,dV$$

$$= \int_0^{2\pi}\int_0^1\int_0^2 r^2z^3\,dz\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1\int_0^2 r^3z^3\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 \left[\frac{r^3z^4}{4}\right]_0^2 dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 \left(\frac{r^3 2^4}{4} - \frac{r^3 0^4}{4}\right) dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 4r^3\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[r^4\right]_0^1 d\theta = \int_0^{2\pi} d\theta$$

$$= \left[\theta\right]_0^{2\pi} = \boxed{2\pi}$$

Good Luck!

1. (25 points) Consider the integral

$$\int_0^1 \int_x^{\sqrt{2-x^2}} (x + 2y)\, dy\, dx.$$

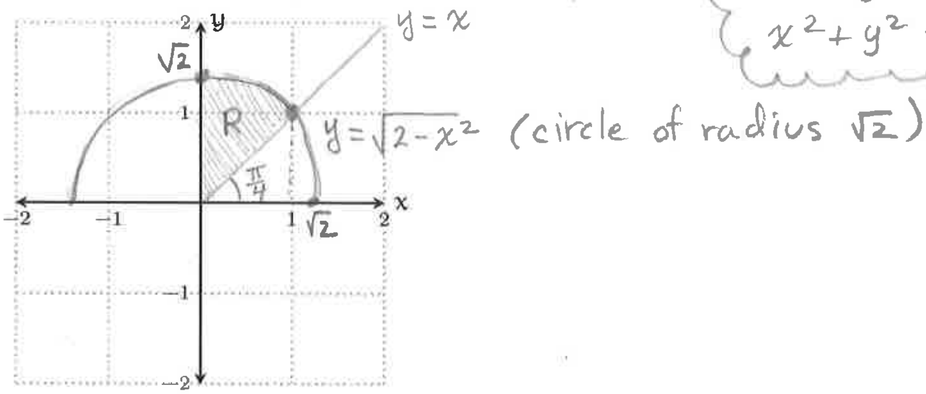
$y = \sqrt{2 - x^2}$

$y^2 = 2 - x^2$

$x^2 + y^2 = 2$

$x^2 + y^2 = \sqrt{2}^2$ ← circle of radius $\sqrt{2}$

(a) Sketch the region of integration.



(b) Convert the integral to a polar integral.

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\sqrt{2}} (r\cos\theta + 2r\sin\theta)\, r\, dr\, d\theta$$

(c) Evaluate your answer from part (b).

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\sqrt{2}} r^2\cos\theta + 2r^2\sin\theta \;\, dr\, d\theta$$

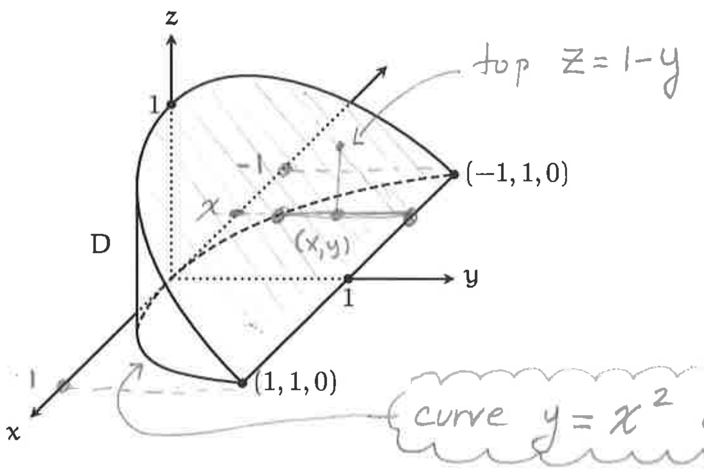
$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left[\frac{r^3}{3}\cos\theta + \frac{2}{3}r^3 \sin\theta \right]_0^{\sqrt{2}} d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left[\frac{r^3}{3}(\cos\theta + 2\sin\theta) \right]_0^{\sqrt{2}} d\theta$$

$$= \int_{\pi/4}^{\pi/2} \frac{2\sqrt{2}}{3}(\cos\theta - 2\sin\theta)\, d\theta$$

$$= \frac{2\sqrt{2}}{3}\Big[\sin\theta - 2\cos\theta\Big]_{\pi/4}^{\pi/2} = \frac{2\sqrt{2}}{3}\left(\left(\sin\frac{\pi}{2} - 2\cos\frac{\pi}{2}\right) - \left(\sin\frac{\pi}{4} - 2\cos\frac{\pi}{4}\right)\right)$$

$$= \frac{2\sqrt{2}}{3}\left((1-0) - \left(\frac{\sqrt{2}}{2} - 2\frac{\sqrt{2}}{2}\right)\right) = \frac{2\sqrt{2}}{3}\left(1 + \frac{\sqrt{2}}{2}\right) = \boxed{\frac{2\sqrt{2}+2}{3}}$$

2. (25 pts.) Consider the region D bounded by the xy-plane, the graph of $y = x^2$, and the plane $y + z = 1$.



For any point (x, y, z) in D,

$$-1 \le x \le 1$$
$$x^2 \le y \le 1$$
$$0 \le z \le 1-y$$

(a) Set up a triple integral for the volume of D.

$$\int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{1-y} dz\, dy\, dx$$

(b) Evaluate the integral to get the volume.

$$= \int_{-1}^{1}\int_{x^2}^{1} \left[z \right]_0^{1-y} dy\, dx$$

$$= \int_{-1}^{1}\int_{x^2}^{1} (1-y)\, dy\, dx$$

$$= \int_{-1}^{1} \left[y - \frac{y^2}{2} \right]_{x^2}^{1} dx$$

$$= \int_{-1}^{1} \left(\left(1 - \frac{1^2}{2}\right) - \left(x^2 - \frac{(x^2)^2}{2}\right) \right) dx$$

$$= \int_{-1}^{1} \left(\frac{1}{2} - x^2 + \frac{x^4}{2} \right) dx = \left[\frac{1}{2}x - \frac{x^3}{3} + \frac{x^5}{10} \right]_{-1}^{1}$$

$$= \left(\frac{1}{2} - \frac{1}{3} + \frac{1}{10} \right) - \left(\frac{1}{2}(-1) - \frac{(-1)^3}{3} + \frac{(-1)^5}{10} \right)$$

$$= \frac{1}{2} - \frac{1}{3} + \frac{1}{10} + \frac{1}{2} - \frac{1}{3} + \frac{1}{10} = 1 - \frac{2}{3} + \frac{1}{5} = \frac{15}{15} - \frac{10}{15} + \frac{3}{15} = \boxed{\frac{8}{15} \text{ cubic units}}$$

3. (25 pts.) Find the average value of the function $f(x, y) = \sin(x + y)$ on the rectangle $0 \leqslant x \leqslant \pi$, $0 \leqslant y \leqslant \frac{\pi}{2}$.

$$\text{Ave} = \frac{\iint_R \sin(x+y)\, dA}{\text{Area of } R}$$

$$= \frac{\int_0^{\pi} \int_0^{\frac{\pi}{2}} \sin(x+y)\, dy\, dx}{(\pi)\left(\frac{\pi}{2}\right)} = \frac{\int_0^{\pi} \left[-\cos(x+y)\right]_0^{\frac{\pi}{2}} dx}{\frac{\pi^2}{2}}$$

$$= \frac{\int_0^{\frac{\pi}{2}} \left(-\cos\left(x + \frac{\pi}{2}\right) + (\cos x + 0)\right) dx}{\frac{\pi^2}{2}}$$

$$= \frac{\left[-\sin\left(x + \frac{\pi}{2}\right) + \sin x\right]_0^{\frac{\pi}{2}}}{\frac{\pi^2}{2}}$$

$$= \frac{\left(-\sin\left(\frac{\pi}{2} + \frac{\pi}{2}\right) + \sin\frac{\pi}{2}\right) - \left(-\sin\left(0 + \frac{\pi}{2}\right) + \sin 0\right)}{\frac{\pi^2}{2}}$$

$$= \frac{-\sin\pi + \sin\frac{\pi}{2} + \sin\frac{\pi}{2} - \sin 0}{\frac{\pi^2}{2}}$$

$$= \frac{0 + 1 + 1 - 0}{\frac{\pi^2}{2}} = \frac{2}{\frac{\pi^2}{2}} = \boxed{\frac{4}{\pi^2}}$$