VCU

MATH 307

MULTIVARIATE CALCULUS

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SAMPLE TEST 2

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Score: 100

Directions. Solve the following questions in the space provided. Unless noted otherwise, you must show your work to receive full credit. This is a closed-book, closed-notes test. Calculators, computers, etc., are not used. Put a your final answer in a $\boxed{\text{box,}}$ where appropriate.

6. (10 pts.) Find the equation of the tangent plane to $f(x,y) = 2x^4 - xy^2 + 3y^2$ at the point $(1,1,4)$.

$f_x(x,y) = 8x^3 - y^2$

$f_y(x,y) = -2xy + 6y$

Tangent plane

$z = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1)$

$z = 4 + 7(x-1) + 4(y-1)$

$z = 4 + 7x - 7 + 4y - 4$

$\boxed{z = 7x + 4y - 7}$

GOOD LUCK!

1. (25 points) Consider the function $z = f(x,y) = xy - x$.

(a) What is the domain of f?

The entire xy-plane

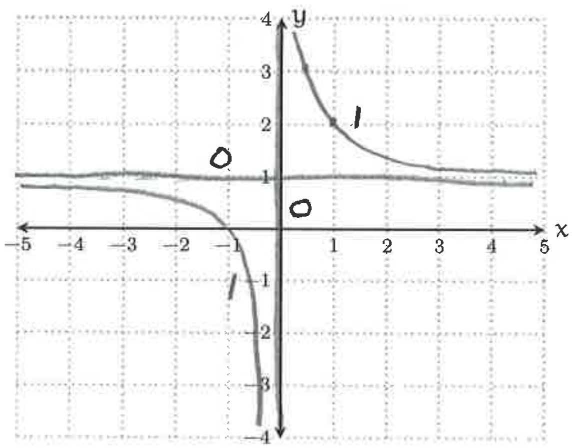
(b) Sketch the level curves for $z = 1$ and $z = 0$.

$z=1$: $xy - x = 1 \rightsquigarrow x(y-1) = 1 \longrightarrow y - 1 = \frac{1}{x}$

$y = \frac{1}{x} + 1$

$z=0$: $xy - x = 0 \rightsquigarrow x(y-1) = 0$

$x = 0 \qquad y = 1$



(c) $\nabla f(x,y) =$

$\langle y-1, x \rangle$

(d) Find the rate of change of $f(x,y)$ in the direction of $\langle 3,5 \rangle$ at the point $(7,3)$.

Unit vector in direction of $\langle 3,5 \rangle$ is $\left\langle \frac{3}{\sqrt{34}}, \frac{5}{\sqrt{34}} \right\rangle$

Rate of change of $f(x,y)$ in direction of $\vec{u} = \left\langle \frac{3}{\sqrt{34}}, \frac{5}{\sqrt{34}} \right\rangle$

is $D_{\vec{u}} f(7,3) = \nabla f(7,3) \cdot \left\langle \frac{3}{\sqrt{34}}, \frac{5}{\sqrt{34}} \right\rangle = \langle 2, 7 \rangle \cdot \left\langle \frac{3}{\sqrt{34}}, \frac{5}{\sqrt{34}} \right\rangle$

$= \boxed{\frac{41}{\sqrt{34}}}$

2. (20 pts.) Evaluate each limit, if possible; if not, explain why it does not exist.

(a) $\lim_{(x,y)\to(0,0)} \frac{5x^3 - 5y^2x}{x^2 - yx}$ $= \lim_{(x,y)\to(0,0)} \frac{5x(x^2-y^2)}{x(x-y)} = \lim_{(x,y)\to(0,0)} \frac{5x(x-y)(x+y)}{x(x-y)}$

$= \lim_{(x,y)\to(0,0)} 5(x+y) = 5(0+0) = \boxed{0}$

(b) $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 - 2y^2}$

Along x-axis $(y=0)$: $\lim_{(x,y)\to(0,0)} \frac{x \cdot 0}{x^2 - 2\cdot 0} = \lim_{(x,y)\to(0,0)} 0 = \boxed{0}$

Along line $x = y$: $\lim_{(x,y)\to(0,0)} \frac{xx}{x^2 - 2x^2} = \lim_{(x,y)\to(0,0)} -1 = \boxed{-1}$

Since we get different values along different paths, limit DNE

3. (15 pts.) Find the maximum and minimum values (and their locations) of the function $f(x, y) = x^2 + y^2$ subject to the constraint $\frac{x^2}{4} + \frac{y^2}{16} = 1$.

$g(x,y) = \frac{x^2}{4} + \frac{y^2}{16} - 1$

$$\begin{cases} \nabla f(x,y) = \lambda \nabla g(x,y) \\ g(x,y) = 0 \end{cases} \Rightarrow \begin{cases} \langle 2x, 2y \rangle = \lambda \langle \frac{x}{2}, \frac{y}{8} \rangle \\ \frac{x^2}{4} + \frac{y^2}{16} = 1 \end{cases}$$

$$\Rightarrow \begin{cases} 2x = \lambda \frac{x}{2} \\ 2y = \lambda \frac{y}{8} \\ \frac{x^2}{4} + \frac{y^2}{16} = 1 \end{cases} \Rightarrow \begin{cases} 4x = \lambda x & (1) \\ 16y = \lambda y & (2) \\ 4x^2 + y^2 = 16 & (3) \end{cases}$$

- If $x \neq 0$, (1) gives $\lambda = 4$. Then (2) gives $16y = 4y$, so $\boxed{y = 0.}$ Then by (3) we get $4x^2 = 16$, so $\boxed{x = \pm 2}$ $\rightsquigarrow$ Get points $(2,0), (-2,0)$.
- If $x = 0$, (3) gives $y^2 = 16$, or $y = \pm 4$ $\rightsquigarrow$ Get points $(0,4), (0,-4)$

$f(2,0) = 2^2 + 0^2 = 4$
$f(-2,0) = (-2)^2 + 0^2 = 4$
$f(0,4) = 0^2 + 4^2 = 16$
$f(0,-4) = 0^2 + (-4)^2 = 16$

$\Rightarrow$ Maximum of 16 at $(0,4)$ and $(0,-4)$
Minimum of 4 at $(2,0)$ and $(-2,0)$

4. (15 pts.) Suppose $f(x,y)$ is a function for which $\nabla f(15,2) = \langle 6, -3\rangle$. Suppose $g(t) = f(t^2-1,\ \sqrt{t})$. Find $g'(4)$.

Note $\nabla f(15,2) = \langle f_x(15,2), f_y(15,2)\rangle = \langle 6,-3\rangle$, so $\begin{cases} f_x(15,2) = 6 \\ f_y(15,2) = -3 \end{cases}$

Now $g(t) = f(x,y)$ where $\begin{cases} x = t^2-1 \\ y = \sqrt{t} \end{cases}$

By chain rule, $g'(t) = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$

$$= f_x(x,y)2t + f_y(x,y)\frac{1}{2\sqrt{t}}$$

$$= f_x(t^2-1, \sqrt{t})2t + f_y(t^2-1, \sqrt{t})\frac{1}{2\sqrt{t}}$$

Then $g'(4) = f_x(4^2-1, \sqrt{4})2\cdot 4 + f_y(4^2-1, \sqrt{4})\dfrac{1}{2\sqrt{4}}$

$$= f_x(15,2)\cdot 8 + f_y(15,2)\frac{1}{4} = 6\cdot 8 - 3\cdot\frac{1}{4} = 48 - \frac{3}{4} = \boxed{\frac{189}{4}}$$

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5. (15 pts.) Consider $f(x,y) = \dfrac{x^3}{3} - x + y^2$.

Find all critical points; classify them as local maxima, local minima or saddle points.

$\nabla f(x,y) = \langle x^2-1, 2y\rangle = \langle (x-1)(x+1), 2y\rangle = \langle 0,0\rangle$

Therefore crit pts. are $(1,0)$ and $(-1,0)$

$f_{xx}(x,y) = 2x$

$f_{yy}(x,y) = 2$

$f_{xy}(x,y) = 0$

Point $(1,0)$: $f_{xx}(1,0)f_{yy}(1,0) - f_{xy}(1,0)^2 = 2\cdot 2 - 0^2 = 4 > 0$

and $f_{xx}(1,0) = 2\cdot 1 > 0$

$\Rightarrow$ $\boxed{\text{Local min at } (1,0)}$

Point $(-1,0)$: $f_{xx}(-1,0)f_{yy}(-1,0) - f_{xy}(-1,0)^2 = -2\cdot 2 - 0^2 = -4 < 0$

$\Rightarrow$ $\boxed{\text{Saddle point at } (-1,0)}$