VCU

MATH 307

MULTIVARIATE CALCULUS

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TEST 1

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Name: Richard

Score: ______

Directions. Solve the following questions in the space provided. Unless noted otherwise, you must show your work to receive full credit. This is a closed-book, closed-notes test. Calculators, computers, etc., are not used. Put a your final answer in a $\boxed{\text{box,}}$ where appropriate.

6. (10 pts.) Consider the following vectors:

$\mathbf{u} = \langle 1, 3, -2 \rangle$, $\mathbf{v} = \langle 2, 2, 4 \rangle$, and $\mathbf{w} = \left\langle -1, -2, \frac{3}{2} \right\rangle$.

State all pairs that are orthogonal to each other.

$\vec{u} \cdot \vec{v} = \boxed{0}$

$\vec{u} \cdot \vec{w} = -1 - 6 - 3 = -10 \neq 0$

$\vec{v} \cdot \vec{w} = -2 - 4 + 6 = \boxed{0}$

Therefore $\boxed{\vec{u} \perp \vec{v} \text{ and } \vec{v} \perp \vec{w}}$

GOOD LUCK!

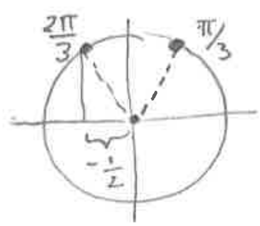
1. (25 points) Let $\mathbf{u} = \langle 1, 1, 0 \rangle$ and $\mathbf{v} = \langle 0, -1, 1 \rangle$.

(a) $|\mathbf{u}| = \sqrt{1^2 + 1^2 + 0} = \boxed{\sqrt{2}}$

(b) Find a unit vector with the same direction as **u**. $\frac{1}{\sqrt{2}} \langle 1, 1, 0 \rangle = \boxed{\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right\rangle}$

(c) $\mathbf{u} \cdot \mathbf{v} = 0 - 1 + 0 = \boxed{-1}$

(d) Find the angle θ between **u** and **v**. $\theta = \cos^{-1}\left(\frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|} \right) = \cos^{-1}\left(\frac{-1}{\sqrt{2}\sqrt{2}} \right) = \cos^{-1}\left(\frac{-1}{2} \right) = \boxed{\frac{2\pi}{3}}$



(e) Find a vector orthogonal to both **u** and **v**.

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{vmatrix} = \boxed{\langle 1, -1, -1 \rangle}$$

2. (20 pts.) Consider the vectors $\mathbf{u} = \langle 1, 1, 3 \rangle$ and $\mathbf{v} = \langle -1, 2, 1 \rangle$ (in standard position).

(a) Find the area of the parallelogram formed by $\mathbf{u}$ and $\mathbf{v}$.

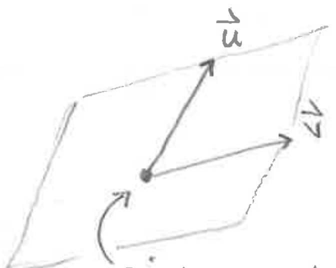
$$A = |\vec{u} \times \vec{v}| = |\langle -5, -4, 3 \rangle| = \sqrt{(-5)^2 + (-4)^2 + 3^2}$$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 3 \\ -1 & 2 & 1 \end{vmatrix} = \langle -5, -4, 3 \rangle$$

$$= \sqrt{25 + 16 + 9} = \sqrt{50}$$

$$= \sqrt{25 \cdot 2} = \boxed{5\sqrt{2} \text{ square units}}$$

(b) Find the equation of the plane that $\mathbf{u}$ and $\mathbf{v}$ line in.



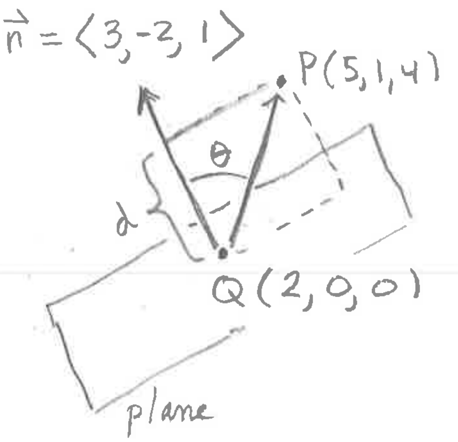
Point on plane: (0, 0, 0)

Plane has normal vector $\vec{n} = \langle -5, -4, 3 \rangle$ (from above) and contains the origin $(0,0,0)$

Equation: $\boxed{-5x - 4y + 3z = 0}$

3. (15 pts.) Find the distance between the point P(5, 1, 4) and the plane whose equation is $3x - 2y + z = 6$.

Normal to the plane is $\vec{n} = \langle 3, -2, 1 \rangle$
Point on plane: $Q(2, 0, 0)$ (because it satisfies $3x-2y+z=6$)



By trigonometry, distance is

$$d = |\overrightarrow{QP}| \cos\theta$$

$$= \frac{|\overrightarrow{QP}|\,|\vec{n}| \cos\theta}{|\vec{n}|}$$

$$= \frac{\overrightarrow{QP} \cdot \vec{n}}{|\vec{n}|} = \frac{\langle 3, 1, 4 \rangle \cdot \langle 3, -2, 1 \rangle}{\sqrt{3^2 + 2^2 + 1}}$$

$$= \frac{9 - 2 + 4}{\sqrt{14}} = \boxed{\frac{11}{\sqrt{14}} \text{ units}}$$

4. (15 pts.) Find the length of the curve
$\mathbf{r}(t) = \left\langle\ t,\ 1,\ \frac{2}{3}t^{3/2}\ \right\rangle$ for $0 \leqslant t \leqslant 8$.

$$\int_0^8 \sqrt{\left(\frac{d}{dt}[t]\right)^2 + \left(\frac{d}{dt}[1]\right)^2 + \left(\frac{d}{dt}\left[\frac{2}{3}t^{3/2}\right]\right)^2}\, dt$$

$$= \int_0^8 \sqrt{1^2 + 0^2 + (t^{1/2})^2}\, dt = \int_0^8 \sqrt{1 + t}\, dt = \int_0^8 (1+t)^{1/2}\, dt$$

$u = 1+t$
$du = dt$
$\int (1+t)^{1/2} dt = \int u^{1/2} du = \frac{2}{3}u^{3/2}$
$= \frac{2}{3}(1+t)^{3/2}$

$$= \left[\frac{2}{3}(1+t)^{3/2}\right]_0^8 = \left[\frac{2}{3}\sqrt{1+t}^{\,3}\right]_0^8$$

$$= \frac{2}{3}\sqrt{1+8}^{\,3} - \frac{2}{3}\sqrt{1+0^2}^{\,3} = \frac{2}{3}\sqrt{9}^{\,3} - \frac{2}{3}\sqrt{1}^{\,3} = \frac{2}{3}3^3 - \frac{2}{3} = \boxed{\frac{52}{3} \text{ units}}$$

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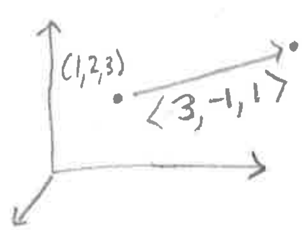
5. (15 pts.) At time $t = 0$ (seconds) a particle is at the point $(1,2,3)$. It travels in a straight line to the point $(4,1,4)$. It has a speed of 2 units per second at $(1,2,3)$ and a constant acceleration of $\langle 3,-1,1\rangle$. Find the position vector $\mathbf{r}(t)$ of the particle.



Note: It travels in the direction of <3,-1,1>

Know: $\vec{r}(0) = (1,2,3)$

$$\vec{v}(0) = 2\frac{\langle 3,-1,1\rangle}{|\langle 3,-1,1\rangle|} = 2\frac{\langle 3,-1,1\rangle}{\sqrt{3^2+1^2+1^2}} = \left\langle \frac{6}{\sqrt{11}}, \frac{-2}{\sqrt{11}}, \frac{2}{\sqrt{11}} \right\rangle$$

$\vec{a}(t) = \langle 3,-1,1\rangle$

$$\vec{v}(t) = \int \vec{a}(t)\,dt = \langle 3t + C_1,\ -t + C_2,\ t + C_3\rangle$$ need to find C's

$$\left\langle \frac{6}{\sqrt{11}}, \frac{-2}{\sqrt{11}}, \frac{2}{\sqrt{11}} \right\rangle = \vec{v}(0) = \langle 3\cdot 0 + C_1,\ -0 + C_2,\ 0 + C_3\rangle = \langle C_1, C_2, C_3\rangle$$

Therefore $\boxed{\vec{v}(t) = \left\langle 3t + \frac{6}{\sqrt{11}},\ -t - \frac{2}{\sqrt{11}},\ t + \frac{2}{\sqrt{11}} \right\rangle}$

Thus $\vec{r}(t) = \int \vec{v}(t)\,dt = \left\langle \frac{3t^2}{2} + \frac{6}{\sqrt{11}}t + C_1,\ -\frac{t^2}{2} - \frac{2}{\sqrt{11}}t + C_2,\ \frac{t^2}{2} + \frac{2}{\sqrt{11}}t + C_3 \right\rangle$

But $\langle 1,2,3\rangle = \vec{r}(0) = \langle C_1, C_2, C_3\rangle$

Thus $\boxed{\vec{r}(t) = \left\langle \frac{3t^2}{2} + \frac{6}{\sqrt{11}}t + 1,\ -\frac{t^2}{2} - \frac{2}{\sqrt{11}} + 2,\ \frac{t^2}{2} + \frac{2}{\sqrt{11}}t + 3 \right\rangle}$