VCU

MATH 307

Multivariate Calculus

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Test 3

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Name: Richard

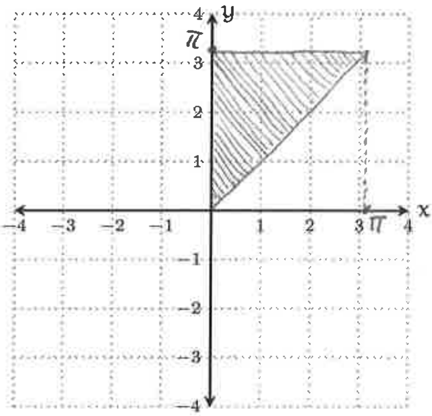
Score: 100

Directions. Answer the questions in the space provided. Unless noted otherwise, you must show and explain your work to receive full credit. Put your final answer in a $\boxed{\text{box}}$ when appropriate.

This is a closed-book, closed-notes test. Calculators, computers, etc., are not used.

1. (20 pts.) This question concerns the integral $\displaystyle\int_0^{\pi}\int_x^{\pi} \frac{\sin(y)}{y}\, dy\, dx.$

(a) Sketch the region of integration on the coordinate axis below.



(b) Rewrite the integral with its order of integration reversed.

$$\int_0^{\pi}\int_0^{y} \frac{\sin(y)}{y}\, dx\, dy$$

(c) Evaluate the integral from part (b) above.

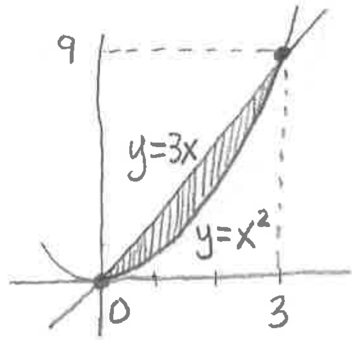
$$\int_0^{\pi}\int_0^{y} \frac{\sin(y)}{y}\, dx\, dy = \int_0^{\pi} \left[\frac{\sin(y)}{y} x\right]_0^{y} dy$$

$$= \int_0^{\pi} \left(\frac{\sin y}{y} y - \frac{\sin y}{y} 0\right) dy$$

$$= \int_0^{\pi} \sin y\, dy = \Big[-\cos y\Big]_0^{\pi}$$

$$= -\cos(\pi) - (-\cos(0)) = 1 + 1 = \boxed{2}$$

2. (20 pts.) Evaluate $\iint_R \frac{8y}{x}\, dA$, where R is the region on the xy-plane between the graphs of $y = x^2$ and $y = 3x$.



$$\iint_R \frac{8y}{x}\, dA = \int_0^3 \int_{x^2}^{3x} \frac{8y}{x}\, dy\, dx$$

$$= \int_0^3 \left[\frac{4y^2}{x}\right]_{x^2}^{3x} dx = \int_0^3 \left(\frac{4(3x)^2}{x} - \frac{4(x^2)^2}{x}\right) dx$$

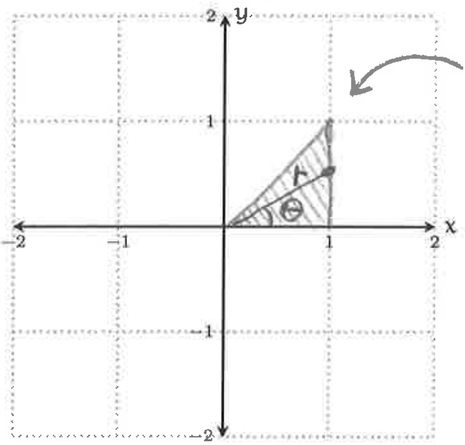
$$= \int_0^3 (36x - 4x^3)\, dx = \left[18x^2 - x^4\right]_0^3$$

$$= 18 \cdot 3^2 - 3^4 = 3^2(18 - 3^2)$$

$$= 9 \cdot 9 = \boxed{81}$$

3. (20 pts.) This question concerns the integral $\int_0^1 \int_0^x \frac{y}{x\sqrt{x^2+y^2}}\, dy\, dx$.

(a) Sketch the region of integration on the coordinate axis below.



Region shaded

Note $0 \le \theta \le \frac{\pi}{4}$

Also $r = \frac{r}{1} = \frac{HYP}{ADJ} = \sec\theta$

(b) Rewrite the integral so that it is in polar from.

$$\int_0^{\frac{\pi}{4}} \int_0^{\sec\theta} \frac{r\sin\theta}{r\cos\theta\sqrt{(r\cos\theta)^2+(r\sin\theta)^2}}\, r\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \int_0^{\sec\theta} \frac{\sin\theta}{\cos\theta\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}}\, r\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \int_0^{\sec\theta} \frac{r}{\sqrt{r^2(\cos^2\theta+\sin^2\theta)}}\, dr\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \int_0^{\sec\theta} \tan\theta \frac{r}{\sqrt{r^2\cdot 1}}\, dr\,d\theta = \boxed{\int_0^{\frac{\pi}{4}} \int_0^{\sec\theta} \tan\theta\, dr\,d\theta}$$

(c) Evaluate the integral.

$$\int_0^{\frac{\pi}{4}} \int_0^{\sec\theta} \tan\theta \, dr \, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \Big[\tan(\theta) r\Big]_0^{\sec\theta} d\theta$$

$$= \int_0^{\frac{\pi}{4}} \tan\theta \sec\theta \, d\theta$$

$$= \Big[\sec\theta\Big]_0^{\frac{\pi}{4}}$$

$$= \sec\frac{\pi}{4} - \sec 0$$

$$= \frac{1}{\cos\frac{\pi}{4}} - \frac{1}{\cos 0} = \frac{1}{\frac{1}{\sqrt{2}}} - \frac{1}{1}$$

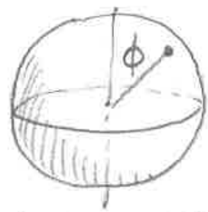
$$= \boxed{\sqrt{2} - 1}$$

4. (20 pts.) Let D be the sphere of radius 2 centered at the origin. We know from geometry that its volume is $V = \frac{4}{3}\pi 2^3 = \frac{32\pi}{3}$.

Compute this volume by evaluating the integral $V = \iiint_D dV$.

You may find spherical coordinates most convenient.

$$V = \iiint_D dV$$



Sphere:
$0 \le \theta \le 2\pi$
$0 \le \phi \le \pi$
$0 \le \rho \le 2$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{2} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \left[\frac{\rho^3}{3}\sin\phi\right]_0^2 d\phi \, d\theta$$

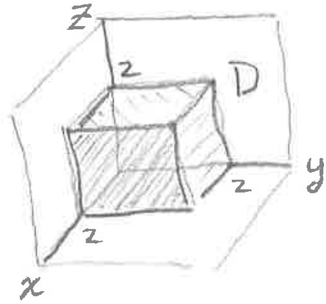
$$= \int_0^{2\pi}\int_0^{\pi} \frac{8}{3}\sin\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \left[-\frac{8}{3}\cos\phi\right]_0^{\pi} d\theta$$

$$= \int_0^{2\pi} \left(-\frac{8}{3}\cos\pi - \left(-\frac{8}{3}\cos 0\right)\right) d\theta$$

$$= \int_0^{2\pi} \frac{16}{3} \, d\theta = \left[\frac{16}{3}\theta\right]_0^{2\pi} = \boxed{\frac{32\pi}{3}}$$

5. (20 pts.) Suppose D is the cube in the first octant, bounded by the three coordinate planes, and the planes $x = 2$, $y = 2$, and $z = 2$. Find the average value of $f(x, y, z) = 3x^2$ on D.



Average value $=$

$$= \frac{\iiint_D 3x^2\, dV}{(\text{Volume of } D)}$$

$$= \frac{\int_0^2 \int_0^2 \int_0^2 3x^2\, dx\, dy\, dz}{2 \cdot 2 \cdot 2}$$

$$= \frac{1}{8} \int_0^2 \int_0^2 \left[x^3\right]_0^2 dy\, dz$$

$$= \frac{1}{8} \int_0^2 \int_0^2 8\, dy\, dz$$

$$= \int_0^2 \int_0^2 dy\, dz = \int_0^2 \left[y\right]_0^2 dz$$

$$= \int_0^2 2\, dz = \left[2z\right]_0^2 = \boxed{4}$$