VCU

MATH 307

Multivariate Calculus

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Test 2

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Score: 100

Directions. Answer the questions in the space provided. Unless noted otherwise, you must show and explain your work to receive full credit. Put your final answer in a $\boxed{\text{box}}$ when appropriate.

This is a closed-book, closed-notes test. Calculators, computers, etc., are not used.

1. (16 pts.) This question concerns the function $f(x, y) = \dfrac{\sqrt{x}}{y-2}$.

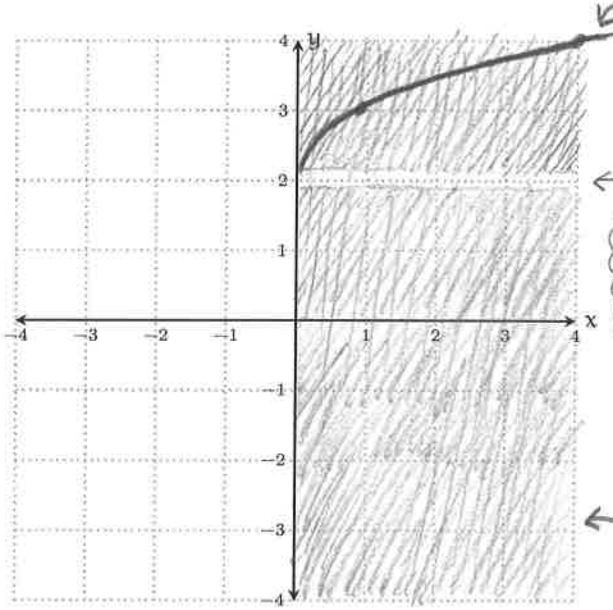
(a) Sketch the domain of this function on the coordinate axis below.

Must have $x \geq 0$ and $y \neq 2$.
All points (x, y) meeting these conditions are shaded below

(b) Using the same coordinate axis, sketch the level curve for $f(x, y) = 1$.

$$\frac{\sqrt{x}}{y-2} = 1 \Rightarrow \sqrt{x} = y - 2 \Rightarrow \boxed{y = \sqrt{x} + 2}$$

This curve is sketched here



line $y=2$ not included in domain

Domain shaded:
All points (x, y)
with $x \geq 0$ and
$y \neq 2$

2. (16 pts.) Suppose $f(x, y) = x^2 - xy + y^2 - y$.

(a) $\nabla f(x, y) = \langle 2x - y, -x + 2y - 1 \rangle$

(b) $\nabla f(1, -1) = \langle 3, -4 \rangle$

(c) Given the unit vector $\mathbf{u} = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$, compute $D_{\mathbf{u}} f(1, -1)$.

$$D_u f(1,-1) = \nabla f(1,-1) \cdot \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$$
$$= \langle 3, -4 \rangle \cdot \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$$
$$= \boxed{\frac{3 - 4\sqrt{3}}{2}}$$

(d) State a unit vector $\mathbf{u}$ for which $D_{\mathbf{u}} f(1, -1)$ is largest.

That would be the unit vector in the direction of $\nabla f(1,-1)$, i.e. $\frac{\langle 3, -4 \rangle}{|\langle 3, -4 \rangle|} = \frac{\langle 3, -4 \rangle}{\sqrt{3^2 + (-4)^2}} = \boxed{\left\langle \frac{3}{5}, -\frac{4}{5} \right\rangle}$

(e) State a unit vector $\mathbf{u}$ for which $D_{\mathbf{u}} f(1, -1) = 0$.

Such a vector is tangen to the level curve at $(1,-1)$, i.e. it is orthogonal to $\nabla f(1,-1) = \langle 3, -4 \rangle$. From part d we therefore get

$$\boxed{\vec{u} = \left\langle \frac{4}{5}, \frac{3}{5} \right\rangle}$$

3. (20 pts.) Find the maximum and minimum values of $x^2 + y^2$ subject to the constraint $x^2 - 2x + y^2 - 4y = 0$.

We want to find max/min of
$f(x,y) = x^2 + y^2$ subject to constraint
$g(x,y) = x^2 - 2x + y^2 - 4y = 0$.

We use the method of Lagrange multipliers.

$$\begin{cases} \nabla f(x,y) = \lambda \nabla g(x,y) \\ g(x,y) = 0 \end{cases}$$

$$\begin{cases} \langle 2x, 2y \rangle = \lambda \langle 2x-2, 2y-4 \rangle \\ x^2 - 2x + y^2 - 4y = 0 \end{cases}$$

$$\begin{cases} \langle x, y \rangle = \lambda \langle x-1, y-2 \rangle \\ x^2 - 2x + y^2 - 4y = 0 \end{cases}$$

$$\begin{cases} x = \lambda x - \lambda & (1) \\ y = \lambda y - 2\lambda & (2) \\ x^2 - 2x + y^2 - 4y = 0 & (3) \end{cases}$$

If $\lambda = 0$, then (1) and (2) give $x = 0$ and $y = 0$, and the system is satisfied.

Get point $(x,y) = (0,0)$

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Now suppose $\lambda \neq 0$. Multiplying ① by y and ② by x yields:

$$\begin{cases} xy = \lambda xy - \lambda y \\ xy = \lambda xy - 2\lambda x \end{cases}$$

Subtracting one from the other,

$$0 = -\lambda y + 2\lambda x$$

Now divide both sides by λ $(\neq 0)$ and transpose:

$$\boxed{y = 2x}$$

Putting this in ③ yields

$$x^2 - 2x + (2x)^2 - 4(2x) = 0$$

$$5x^2 - 10x = 0$$

$$5x(x-2) = 0$$

$x = 0$, $y = 2 \cdot 0 = 0$

$x = 2$, $y = 2 \cdot 2 = 4$

Get points $(0,0)$ and $(2,4)$

$f(0,0) = 0^2 + 0^2 = 0$ ← MIN at $(0,0)$

$f(2,4) = 2^2 + 4^2 = 20$ ← MAX at $(2,4)$

4. (20 pts.) Find the critical points of the function $f(x, y) = xe^y - 5x$.
(Just find the critical points – no need to classify them as local max/min.)

Solve $\nabla f(x,y) = \langle 0, 0 \rangle$

$\langle e^y - 5, xe^y \rangle = \langle 0, 0 \rangle$

$e^y - 5 = 0$

$e^y = 5$

$\ln e^y = \ln 5$

$\boxed{y = \ln 5}$

$xe^y = 0$

Because $e^y > 0$ must have

$\boxed{x = 0}$

Therefore just one critical point and it is $\boxed{(0, \ln 5)}$

5. (12 pts.) Consider $f(x,y) = y + \sin(xy + \pi)$.

(a) $\dfrac{\partial f}{\partial x} = \boxed{\cos(xy+\pi)\,y}$

$= \boxed{y\cos(xy+\pi)}$

(b) $\dfrac{\partial f}{\partial y} = \boxed{1 + \cos(xy+\pi)\,x}$

$= \boxed{1 + x\cos(xy+\pi)}$

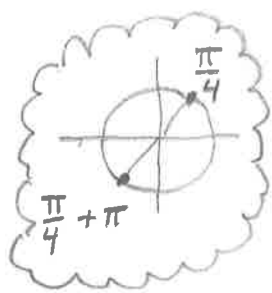
(c) $\dfrac{\partial^2 f}{\partial y \partial x} = \boxed{\cos(xy+\pi) - yx\sin(xy+\pi)}$

(product rule)

(d) $f_x\left(\frac{\pi}{8}, 2\right) = \cos\left(\frac{\pi}{8}\cdot 2 + \pi\right)\cdot 2$

$= 2\cos\left(\frac{\pi}{4} + \pi\right)$

$= 2\left(-\frac{\sqrt{2}}{2}\right) = \boxed{-\sqrt{2}}$



6. (12 pts.) Evaluate the limit or explain why it does not exist.

$$\lim_{(x,y)\to(2,0)} \frac{\sqrt{2x-y}-2}{2x-y-4}$$

Gives $\frac{0}{0}$, so try to cancel

$$= \lim_{(x,y)\to(2,0)} \frac{\sqrt{2x-y}-2}{\sqrt{2x-y}^{\,2}-2^2}$$

$$= \lim_{(x,y)\to(2,0)} \frac{\cancel{\sqrt{2x-y}-2}}{\cancel{(\sqrt{2x-y}-2)}(\sqrt{2x-y}+2)}$$

$$= \lim_{(x,y)\to(2,0)} \frac{1}{\sqrt{2x-y}+2}$$

$$= \frac{1}{\sqrt{2\cdot 2-0}+2} = \frac{1}{2+2}$$

$$= \boxed{\frac{1}{4}}$$