12. Let R be the rectangle $0 \leqslant x \leqslant \ln 2$, $0 \leqslant y \leqslant \ln 2$.

Compute $\displaystyle\iint_R e^{x-y}\,dA$.

$$= \int_0^{\ln 2}\int_0^{\ln 2} e^{x-y}\,dy\,dx$$

$$= \int_0^{\ln 2}\Big[-e^{x-y}\Big]_0^{\ln 2}\,dx$$

$$= \int_0^{\ln 2}\left(-e^{x-\ln 2} + e^{x-0}\right)dx$$

$$= \Big[-e^{x-\ln 2} + e^{x}\Big]_0^{\ln 2}$$

$$= \left(-e^{\ln 2-\ln 2} + e^{\ln 2}\right) - \left(-e^{0-\ln 2} + e^{0}\right)$$

$$= \left(-e^{0} + e^{\ln 2}\right) - \left(-e^{-\ln 2} + e^{0}\right)$$

$$= -1 + 2 + e^{\ln 2^{-1}} - 1$$

$$= -1 + 2 + 2^{-1} - 1$$

$$= 2^{-1} = \boxed{\frac{1}{2}}$$

VCU

MATH 307

MULTIVARIATE CALCULUS

R. Hammack

FINAL EXAM

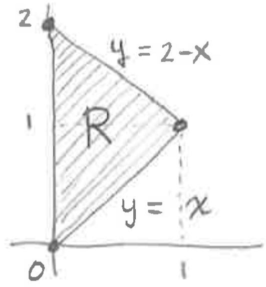
December 10, 2013

Name: Richard

Score: 100

Directions. Solve the questions in the space provided. Unless noted otherwise, you must show your work to receive full credit. This is a closed-book, closed-notes test. Calculators, computers, etc., are not used. Put a your final answer in a box, where appropriate.

1. Compute the mass of a triangular plate bounded by the y-axis, the line $y = x$ and the line $y = 2 - x$, if the plate's density at point (x, y) is $\delta(x, y) = x + 2y$.



$$M = \iint_R \delta(x,y)\, dA$$

$$= \int_0^1 \int_x^{2-x} (x + 2y)\, dy\, dx$$

$$= \int_0^1 \Big[xy + y^2 \Big]_x^{2-x} dx$$

$$= \int_0^1 \left(x(2-x) + (2-x)^2 \right) - \left(xx + x^2 \right) dx$$

$$= \int_0^1 \left(2x - x^2 + 4 - 4x + x^2 - x^2 - x^2 \right) dx$$

$$= \int_0^1 \left(4 - 2x - 2x^2 \right) dx$$

$$= \Big[4x - x^2 - \frac{2}{3} x^3 \Big]_0^1$$

$$= 4 - 1 - \frac{2}{3} = 3 - \frac{2}{3} = \frac{9}{3} - \frac{2}{3} = \boxed{\frac{7}{3}}$$

2. Find all the local maxima, minima and saddle points of the function $f(x, y) = 4 - x^2 - xy - y^2 - 3x + 3y$.

$$\nabla f(x,y) = \langle \underbrace{-2x - y - 3}_{f_x},\ \underbrace{-2y - x + 3}_{f_y} \rangle = \langle 0, 0 \rangle$$

$$\Rightarrow \quad \begin{aligned} -2x - y - 3 &= 0 \\ -2y - x + 3 &= 0 \end{aligned}$$

(add)

$$\begin{aligned} -3y - 3x &= 0 \\ y &= -x \end{aligned}$$

Now put $y = -x$ into $-2x - y - 3 = 0$.

Get $-2x - (-x) - 3 = 0$

$$-x = 3$$

$$x = -3, \quad y = -(-3) = 3$$

Thus only critical point is $(-3, 3)$

$f_{xx} = -2$

$f_{yy} = -2$

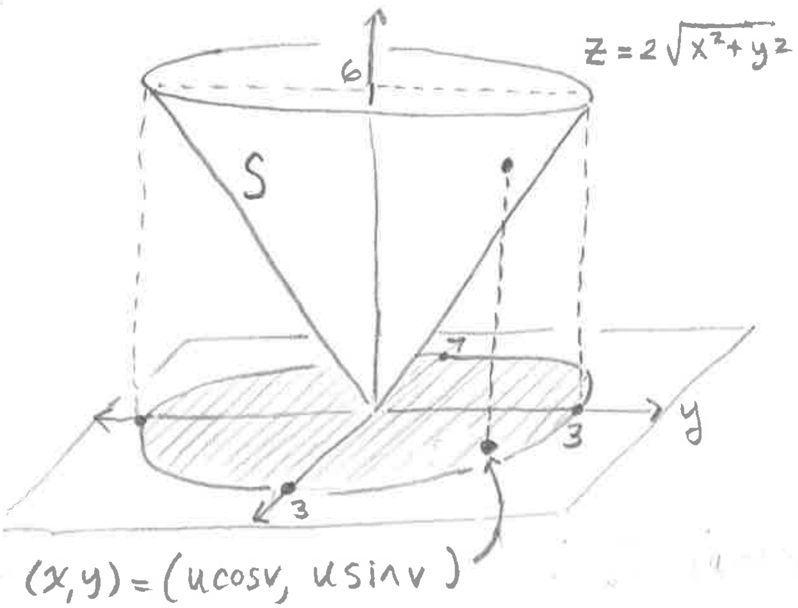
$f_{xy} = -1$

$$f_{xx} f_{yy} - f_{xy}^2 = (-2)(-2) - (-1)^2 = 4 - 1 = 3 > 0$$

Therefore there is either a local max or a local min at $(-3, 3)$. To see which, note that $f_{xx}(-3,3) = -2 < 0$

Therefore there is a $\boxed{\text{local maximum at } (-3, 3)}$

3. Find the area of the part of the surface $z = 2\sqrt{x^2+y^2}$ that lies between the planes $z = 0$ and $z = 6$.



This surface S is a cone that lies above the circle of radius 3 centered at the origin of the xy-plane.

Any point (x, y) on this circle has polar form $(x,y) = (u\cos v, u\sin v)$.

The point on S above (x,y) is $\langle u\cos v, u\sin v, 2\sqrt{(u\cos v)^2+(u\sin v)^2}\rangle$

$= \langle u\cos v, u\sin v, 2u\rangle$

Therefore S is parameterized as $\vec{r}(u,v) = \langle u\cos v, u\sin v, 2u\rangle$ for $0 \le u \le 3$ and $0 \le v \le 2\pi$.

$$\vec{r}_u = \langle \cos v, \sin v, 2\rangle$$

$$\vec{r}_v = \langle -u\sin v, u\cos v, 0\rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ \cos v & \sin v & 2 \\ -u\sin v & u\cos v & 0 \end{vmatrix} = \langle -2u\cos v, -2u\sin v, u\cos^2 v + u\sin^2 v\rangle$$

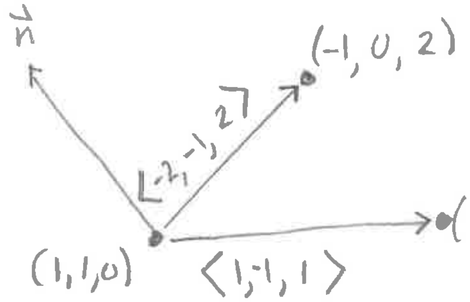
$$= \langle -2u\cos v, -2u\sin v, u\rangle$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{(-2u\cos v)^2 + (-2u\sin v)^2 + u^2} = \sqrt{4u^2+u^2} = \sqrt{5u^2} = \boxed{u\sqrt{5}}$$

$$\underline{\text{AREA}} = \iint_S d\sigma = \int_0^{2\pi}\int_0^3 |\vec{r}_u \times \vec{r}_v|\, du\, dv = \int_0^{2\pi}\int_0^3 u\sqrt{5}\, du\, dv$$

$$= \int_0^{2\pi}\left[\frac{\sqrt{5}}{2}u^2\right]_0^3 dv = \frac{9\sqrt{5}}{2}\int_0^{2\pi} dv = \frac{18\pi\sqrt{5}}{2} = \boxed{9\pi\sqrt{5} \text{ sq. units}}$$

4. Find the equation of the plane through $(1,1,0)$, $(-1,0,2)$ and $(2,0,1)$.



Normal to the plane is

$$\vec{n} = \langle 1,-1,1\rangle \times \langle -2,-1,2\rangle$$

$$= \begin{vmatrix} i & j & k \\ 1 & -1 & 1 \\ -2 & -1 & 2 \end{vmatrix}$$

$$= \langle -1, -4, -3\rangle$$ $\rightsquigarrow$ Thus eqn. has form $-x-4y-3z = D$

Plug in point $(2,0,1)$ to get $-2-4\cdot 0-3\cdot 1 = D \implies D=-5$

Thus equation is $-x-4y-3z=-5$, or $\boxed{x+4y+3z=5}$

5. In what direction is the derivative of the function $f(x,y) = x^2y + y^2x$ at $P(3,2)$ equal to zero? Explain your reasoning.

$$\nabla f(x,y) = \langle 2xy+y^2,\ x^2+2xy\rangle$$

Let $\vec{u} = \langle a, b\rangle$ be a unit vector

At $(3,2)$ the derivative in the direction of $\vec{u}$ is

$$D_u f = \nabla f(3,2)\cdot \vec{u} = \langle 2\cdot 3\cdot 2+2^2,\ 3^2+2\cdot 3\cdot 2\rangle \cdot \langle a,b\rangle$$

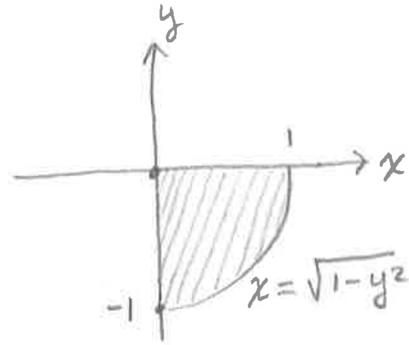
$$= \langle 16, 21\rangle \cdot \langle a,b\rangle$$

$$= 16a + 21b$$

Notice that if $\langle a,b\rangle = \langle 21,-16\rangle$, then the derivative in this direction is zero. But presumably $\vec{u} = \langle a,b\rangle$ is a unit vector, so it is $\vec{u} = \dfrac{\langle 21,-16\rangle}{|\langle 21,-16\rangle|}$

$$= \frac{\langle 21,-16\rangle}{\sqrt{21^2+16^2}} = \boxed{\frac{\langle 21,-16\rangle}{\sqrt{697}}}$$

6. Sketch the region of integration and integrate: $\displaystyle\int_{-1}^{0}\int_{0}^{\sqrt{1-y^2}} \frac{4}{1+x^2+y^2}\,dx\,dy$



The region is a quarter of the unit circle, in the fourth quadrant. →

Converting to polar, any point (x,y) in this region has form $(r\cos\theta,\ r\sin\theta)$, $0 \le r \le 1$, $-\frac{\pi}{2} \le \theta \le 0$

Therefore $\displaystyle\int_{-1}^{0}\int_{0}^{\sqrt{1-y^2}} \frac{4}{1+x^2+y^2}\,dx\,dy = \int_{-\frac{\pi}{2}}^{0}\int_{0}^{1} \frac{4}{1+(r\cos\theta)^2+(r\sin\theta)^2}\,r\,dr\,d\theta$

$$= \int_{-\frac{\pi}{2}}^{0}\int_{0}^{1} \frac{4r}{1+r^2\cos^2\theta+r^2\sin^2\theta}\,dr\,d\theta = \int_{-\frac{\pi}{2}}^{0}\int_{0}^{1} \frac{4r}{1+r^2}\,dr\,d\theta$$

$$= \int_{-\pi/2}^{0} \Big[2\ln(1+r^2)\Big]_0^1\,d\theta = \int_{-\pi/2}^{0} (2\ln 2 - 2\ln 1)\,d\theta = \int_{-\frac{\pi}{2}}^{0} 2\ln 2\,d\theta = \boxed{\pi \ln 2}$$

7. Find the work done by $\mathbf{F}$ over the curve in the direction of increasing t.
$\mathbf{F} = \langle xz, z, y\rangle$ and $\mathbf{r}(t) = \langle t, t^2, t\rangle$, $0 \leqslant t \leqslant 1$.

$$W = \int_C F\cdot T\,ds = \int_C F\cdot \underbrace{\frac{dr}{dt}\frac{1}{|v(t)|}}_{T}\ \underbrace{|v(t)|\,dt}_{ds}$$

$$= \int_C F\cdot\frac{dr}{dt}\,dt = \int_0^1 \langle xz, z, y\rangle\cdot\langle 1, 2t, 1\rangle\,dt$$

$$= \int_0^1 \langle tt, t, t^2\rangle\cdot\langle 1, 2t, 1\rangle\,dt$$

$$= \int_0^1 (t^2+2t^2+t^2)\,dt = \int_0^1 4t^2\,dt =$$

$$= \Big[\frac{4}{3}t^3\Big]_0^1 = \boxed{\frac{4}{3}}$$

8. This problem concerns the vector field
$$F(x,y,z) = \left\langle \frac{1}{y},\ \frac{1}{z} - \frac{x}{y^2},\ -\frac{y}{z^2} \right\rangle.$$

(a) The field **F** is conservative. (You do not need to show this.) Find a potential function for **F**.

We seek a potential function $f(x,y,z)$ for which $F = \nabla f$, i.e.,
$$\left\langle \frac{1}{y}, \frac{1}{z} - \frac{x}{y^2}, -\frac{y}{z^2} \right\rangle = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle.$$

From this, $\frac{\partial f}{\partial x} = \frac{1}{y}$ so $f(x,y,z) = \int \frac{1}{y}\,dx = \frac{x}{y} + g(x,z)$

Thus $\boxed{f(x,y,z) = \frac{x}{y} + g(x,y).}$ We now need to find $g(x,z)$.

Note $\frac{\partial f}{\partial z} = -\frac{y}{z^2} = \frac{\partial g}{\partial z} \Rightarrow g(y,z) = \int -\frac{y}{z^2}\,dz = \frac{y}{z} + h(y)$.

Now we have $g(y,z) = \frac{y}{z} + h(y)$ so $\boxed{f(x,y,z) = \frac{x}{y} + \frac{y}{z} + h(y)}$

Note $\frac{\partial f}{\partial y} = \frac{1}{z} - \frac{x}{y^2} = -\frac{x}{y^2} + \frac{1}{z} + \frac{dh}{dy} \Rightarrow \frac{dh}{dy} = 0$

Therefore $h(y)$ is a constant, which we may set to 0.

Potential Function: $\boxed{f(x,y,z) = \frac{x}{y} + \frac{y}{z}}$

(b) Suppose C is the following curve:
$$\mathbf{r}(t) = \left\langle \frac{3-\cos(\pi t)}{2},\ 1+t^2,\ 2^t \right\rangle \text{ for } 0 \leqslant t \leqslant 1.$$

Use your answer from part **(a)** above to compute
$$\int_C \frac{1}{y}dx + \left(\frac{1}{z} - \frac{x}{y^2}\right) dy - \frac{y}{z^2}dz.$$

This curve begins at point $A = \vec{r}(0) = \langle 1, 1, 1 \rangle$ and ends at $B = \vec{r}(1) = \langle 2, 2, 2 \rangle$

$$= \int_C \left\langle \frac{1}{y}, \frac{1}{z} - \frac{x}{y^2}, \frac{y}{z^2} \right\rangle \cdot \langle dx, dy, dz \rangle$$

$$= \int_C F \cdot dr = f(B) - f(A) = f(2,2,2) - f(1,1,1)$$

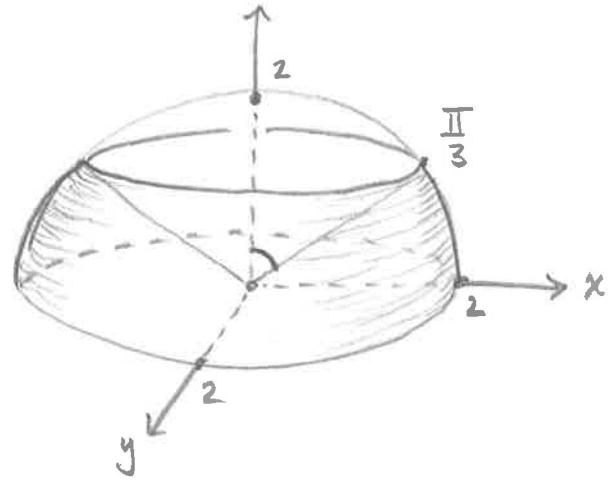
Because $F = \nabla f$ is conservative

$$= \left(\frac{2}{2} + \frac{2}{2}\right) - \left(\frac{1}{1} + \frac{1}{1}\right)$$

$$= 2 - 2 = \boxed{0}$$

9. Using spherical coordinates, set up the triple integral that gives the volume of the solid bounded below by the xy-plane, on the sides by the sphere $\rho = 2$, and above by the cone $\phi = \frac{\pi}{3}$.

Once you have set up the integral, evaluate it.



Note: Any point in this solid has spherical coordinates (ρ, ϕ, θ)

for $0 \le \rho \le 2$

$\frac{\pi}{3} \le \phi \le \frac{\pi}{2}$

$0 \le \theta \le 2\pi$

$$\text{Volume} = \int_0^{2\pi} \int_{\pi/3}^{\pi/2} \int_0^2 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_{\pi/3}^{\pi/2} \left[\frac{\rho^3 \sin\phi}{3} \right]_0^2 d\phi \, d\theta = \int_0^{2\pi} \int_{\pi/3}^{\pi/2} \frac{8}{3} \sin\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \left[-\frac{8}{3} \cos\phi \right]_{\pi/3}^{\pi/2} d\theta = \int_0^{2\pi} \left(-\frac{8}{3} \cos\frac{\pi}{2} + \frac{8}{3} \cos\frac{\pi}{3} \right) d\theta$$

$$= \int_0^{2\pi} \frac{8}{3} \cos\frac{\pi}{3} \, d\theta = \int_0^{2\pi} \frac{8}{3} \frac{1}{2} \, d\theta = \frac{4}{3} \int_0^{2\pi} d\theta$$

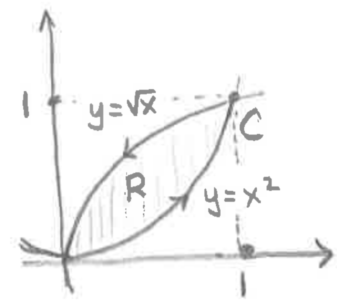
$$= \frac{4}{3} 2\pi = \boxed{\frac{8\pi}{3} \text{ cubic units}}$$

10. Recall that Green's theorem asserts that

$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy.$$

Let C be the curve (traversed counterclockwise) that contains the region R between the graphs of $y = x^2$ and $x = y^2$. Use Green's Theorem to find

$$\oint_C \underbrace{(xy + y^2)}_{M}dx + \underbrace{(x - y)}_{N}dy.$$



Here's a picture of C and the enclosed region R

$$= \oint_C M dx + N dy.$$

$$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy = \iint_R (1 - (x + 2y))\, dx\, dy$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} (1 - x - 2y)\, dy\, dx$$ ← switch dx & dy to better fit region R

$$= \int_0^1 \left[y - xy - y^2\right]_{x^2}^{\sqrt{x}} dx = \int_0^1 \left(\sqrt{x} - x\sqrt{x} - \sqrt{x}^2\right) - \left(x^2 - x x^2 - (x^2)^2\right) dx$$

$$= \int_0^1 \left(x^{\frac{1}{2}} - x^{\frac{3}{2}} - x - x^2 + x^3 + x^4\right) dx$$

$$= \left[\frac{2}{3}x^{3/2} - \frac{2}{5}x^{\frac{5}{2}} - \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5}\right]_0^1$$

$$= \left[\frac{2}{3}\sqrt{x}^3 - \frac{2}{5}\sqrt{x}^5 - \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5}\right]_0^1$$

$$= \frac{2}{3} - \frac{2}{5} - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{1}{3} - \frac{1}{5} - \frac{1}{4}$$

$$= \frac{20}{60} - \frac{12}{60} - \frac{15}{60} = \boxed{-\frac{7}{60}}$$

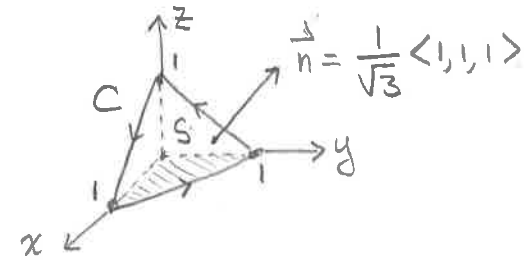
11. Recall that Stokes' theorem asserts that

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \nabla\times\mathbf{F}\cdot\mathbf{n}\,d\sigma.$$

Let C be the boundary of the triangle cut out from the plane $x+y+z=1$ by the first octant, counterclockwise when viewed from above. Let $\mathbf{F}(x,y,z)=\langle y,\ xz,\ x^2\rangle$.

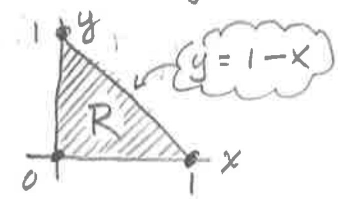
Use Stokes' theorem to compute $\oint_C \mathbf{F}\cdot d\mathbf{r}$.

First, let's draw the curve C



It is the boundary of a triangle S
The unit normal to the triangle S is $\vec{n}=\dfrac{\langle 1,1,1\rangle}{|\langle 1,1,1\rangle|}=\dfrac{1}{\sqrt{3}}\langle 1,1,1\rangle$.

Notice that the surface S is the part of the graph of $z=f(x,y)=1-x-y$ above a triangular region R on the xy-plane, illustrated here:



Note: $\nabla\times F=\begin{vmatrix} i & j & k\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ y & xz & x^2\end{vmatrix}=\langle -x,\ -2x,\ z-1\rangle$

And $\nabla\times F\cdot\vec{n}=\langle -x,-2x,z-1\rangle\cdot\frac{1}{\sqrt{3}}\langle 1,1,1\rangle=\dfrac{-3x+z-1}{\sqrt{3}}$

Finally $\displaystyle\oint_C F\cdot dr=\iint_S \nabla\times F\cdot\vec{n}\,d\sigma=\iint_S \frac{-3x+z-1}{\sqrt{3}}\,d\sigma$

$\displaystyle =\iint_S \frac{-3x+(1-x-y)-1}{\sqrt{3}}\,d\sigma=\iint_S \frac{-4x-y}{\sqrt{3}}\,d\sigma$

$\displaystyle =\iint_R \frac{-4x-y}{\sqrt{3}}\sqrt{f_x^2+f_y^2+1}\,dA=\iint_R \frac{-4x-y}{\sqrt{3}}\sqrt{1^2+1^2+1}\,dA=\iint_R -4x-y\,dA$

$\displaystyle =\int_0^1\int_0^{1-x}(-4x-y)\,dy\,dx=\int_0^1\left[-4xy-\frac{y^2}{2}\right]_0^{1-x}dx=\int_0^1 -4x(1-x)-\tfrac{1}{2}(1-x)^2\,dx$

$\displaystyle =\int_0^1\left(-4x+4x^2-\tfrac{1}{2}+x-\tfrac{1}{2}x^2\right)dx=\int_0^1\left(\tfrac{7}{2}x^2-3x-\tfrac{1}{2}\right)dx$

$\displaystyle =\left[\tfrac{7}{6}x^3-\tfrac{3}{2}x^2-\tfrac{1}{2}x\right]_0^1=\tfrac{7}{6}-\tfrac{3}{2}-\tfrac{1}{2}=\tfrac{7}{6}-\tfrac{4}{2}=\tfrac{7-12}{6}=\boxed{-\tfrac{5}{6}}$