Section 16.7 Stokes' Theorem

Recall that for a v.f. $F = \langle M, N \rangle$ on the plane we have

$\text{div } F = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = \begin{pmatrix}\text{measures compression} \\ \text{or expansion at } (x,y)\end{pmatrix}$

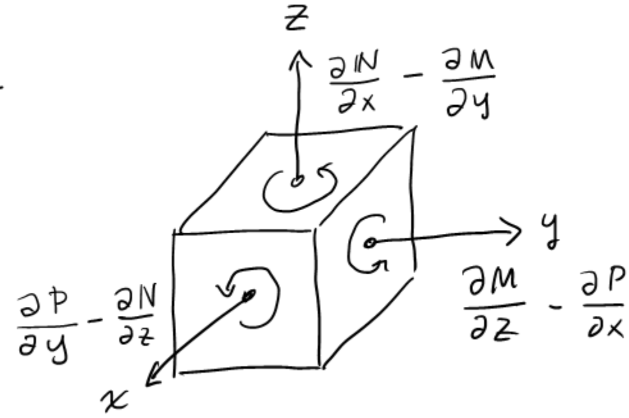
$\text{curl } F = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = \begin{pmatrix}\text{measures counterclockwise} \\ \text{circulation at } (x,y)\end{pmatrix}$

Now lets adapt all this to 3-D. Let $F = \langle M, N, P \rangle$ and think of it as representing the velocity of a fluid or gas flowing in space.

Divergence

$$\text{div } F = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} = \underbrace{\overbrace{\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle}^{\nabla} \cdot \overbrace{\langle M, N, P \rangle}^{F} = \nabla \cdot F}_{\text{special notation}}$$

Curl

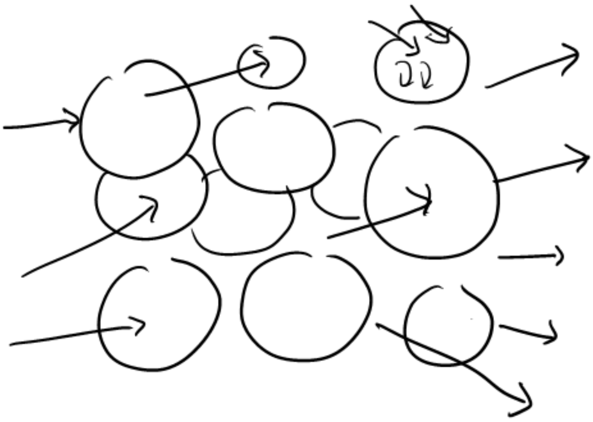


$$\text{curl } F = \left\langle \frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right\rangle = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix} = \nabla \times F$$

= (Axis of spin in v.f. F at point (x, y, z)

|curl F| = speed of spin)

Imagine v.f. F moving little balls through space
As they flow through space, they also may spin.

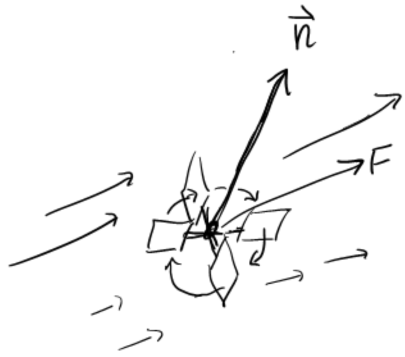


$F(x,y,z)$ = velocity of ball at (x,y,z)

$\text{div } F = \begin{pmatrix}\text{Compression of balls} \\ \text{at } (x,y,z)\end{pmatrix}$

$\text{curl } F = \begin{pmatrix}\text{spin of ball} \\ \text{at } (x,y,z)\end{pmatrix}$

Given a unit vector $\vec{n}$, imagine that it has a little 'paddle' wheel at its end. The v.f. F would cause it to spin.

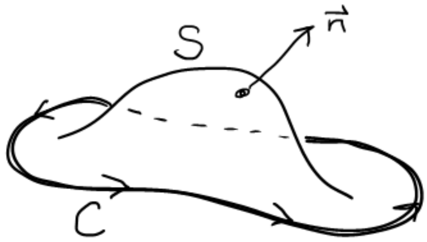


$$(\operatorname{curl} F)\cdot\vec{n} = \nabla\times F\cdot\vec{n} = \begin{cases}\text{a measure of the spin of } \vec{n}.\\ \text{positive: } \circlearrowleft \rightarrow \vec{n}\\ \text{negative: } \circlearrowright \rightarrow \vec{n}\\ \text{zero} - \text{no spin}\\ \text{bigger} = \text{faster spin}\end{cases}$$

With all this in mind, we can state and understand the main result of this section — Stokes' Theorem.

Stoke's Theorem

Suppose $F = \langle M, N, P\rangle$ is a vector field in space and S is a surface with normal $\vec{n}$ and boundary C, a curve $\vec{r}(t)$ traversed counterclockwise (looking down $\vec{n}$). Then:

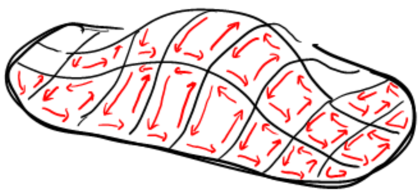


$$\oint_C F\cdot d\vec{r} = \iint_S \nabla\times F\cdot\vec{n}\, d\sigma$$

$\nabla\times F\cdot\vec{n} = (\operatorname{curl} F)\cdot\vec{n}$ = "flux of curl"

i.e. $\begin{pmatrix}\text{circulation}\\ \text{around } C\end{pmatrix} = \iint_S \begin{pmatrix}\text{circulation at point}\\ (x,y,z) \text{ on surface}\end{pmatrix} d\sigma$

You can understand this intuitively by dividing S up into small rectangles



In $\iint_S \nabla F\cdot n\, d\sigma = \lim_{|P|\to 0}\sum \nabla f\cdot n\,\Delta\sigma_k$

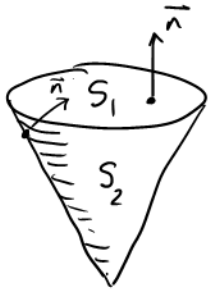
the contributions to circulation along adjacent rectangles cancel. The only contribution is along the boundary C; it is $\int_C F\cdot dr$.

Notice that the theorem implies that if surfaces S_1 and S_2 share a common boundary, then

$$\iint_{S_1} \nabla\times F\cdot\vec{n}\, d\sigma = \iint_{S_2}\nabla\times F\cdot n\, d\sigma$$

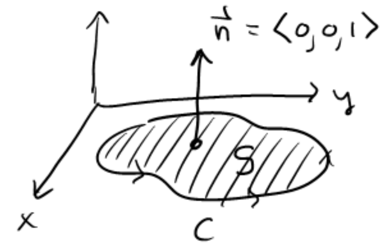
because both sides equal $\oint_C F\cdot dr$

For instance suppose S_1 is top of cone and S_2 is side. Then



$$\iint_{S_1}\nabla\times F\cdot n\, d\sigma = \iint_{S_2}\nabla\times F\cdot n\, d\sigma$$

Notice also what happens if S is a flat surface on the xy-plane



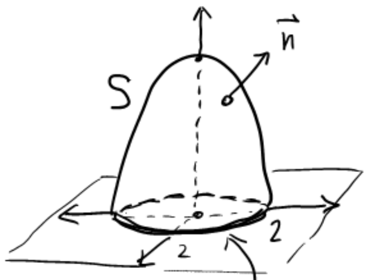
$$\oint_C F \cdot dr = \iint_S (\nabla \times F) \cdot \langle 0,0,1 \rangle \, d\sigma$$

$$\Rightarrow \quad \oint F \cdot T \, ds = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

In this case, Stokes' theorem reduces to Greens' Theorem.
In other words, Stokes theorem is a generalization to Greens Theo..

<u>Example</u> $F(x,y,z) = \langle y, -x^2, 2z^2 \rangle$
S is the part of $4 - x^2 - y^2$ above xy-plane

Verify that Stokes' theorem holds.



Base circle C
of radius 2
is boundary of S
$\vec{r}(t) = \langle 2\cos t, 2\sin t, 0 \rangle$
$0 \le t \le 2\pi$

<u>Left hand side:</u>

$$\oint_C F \cdot dr = \oint_C F \cdot \frac{dr}{dt} \, dt$$

$$= \int_0^{2\pi} \langle 2\sin t, -(2\cos t)^2, 2 0^2 \rangle \cdot \langle -2\sin t, 2\cos t, 0 \rangle dt$$

$$= \int_0^{2\pi} \left(-4\sin^2 t - 8\cos^3 t \right) dt = \int_0 \left(-4 \frac{1 - \cos(2t)}{2} - 8\cos^2 t \cos t \right) dt$$

$$= \int_0^{2\pi} -2(1 - \cos 2t) - 8(1 - \sin^2 t) \cos t \, dt$$

$u = \sin t$
$du = \cos t \, dt$

$$= \int_0^{2\pi} \left(-2 + \cos 2t - 8\cos t + 8(\sin t)^2 \cos t \right) dt$$

$$= \left[-2t + \frac{1}{2}\sin 2t - 8\sin t + 8\frac{\sin^3 t}{3} \right]_0^{2\pi}$$

$$= \left(-4\pi + \frac{1}{2}\sin 4\pi - 8\sin 2\pi + 8\frac{\sin^3 2\pi}{3} \right) - \left(-2 \cdot 0 + \frac{1}{2}\sin 0 - 8\sin 0 + 8\frac{\sin^3 0}{3} \right)$$

$$= \boxed{-4\pi}$$

Now let's compute the right-hand side.

$$\iint_S \nabla \times F \cdot n \, d\sigma.$$

First, $\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x^2 & 2z^2 \end{vmatrix} = \langle 0, 0, -2x-1 \rangle$ ← hold that thought

Next, we parameterize the surface S as:

$$\left.\begin{aligned} \vec{r}(u,v) &= \langle u\cos v,\ u\sin v,\ 4-(u\cos v)^2-(u\sin v)^2 \rangle \\ &= \langle u\cos v,\ u\sin v,\ 4-u^2 \rangle \end{aligned}\right\} \ 0 \le u \le 2, \quad 0 \le v \le 2\pi$$

$\vec{r}_u = \langle \cos v, \sin v, -2u \rangle$

$\vec{r}_v = \langle -u\sin v, u\cos v, 0 \rangle$

$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ \cos v & \sin v & -2u \\ -u\sin v & u\cos v & 0 \end{vmatrix} = \langle 2u^2\cos v, -2u^2\sin v, u \rangle$

not needed

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{(2u^2\cos v)^2 + (-2u^2\sin v)^2 + u^2} = \sqrt{4u^4\cos^2 v + 4u^4\sin^2 v + u^2}$$

$$= \sqrt{4u^4(\cos^2 v + \sin^2 v) + u^2} = \sqrt{4u^4 + u^2} = u\sqrt{4u^2+1}$$

<u>Now</u>: $$\iint_S \nabla \times F \cdot n \, d\sigma = \int_0^{2\pi}\int_0^2 \underbrace{\langle 0,0,-2x-1\rangle}_{\nabla \times F} \cdot \underbrace{\frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}}_{\vec{n}} \underbrace{|r_u \times r_v| \, du\,dv}_{d\sigma}$$

$$= \int_0^{2\pi}\int_0^2 \langle 0,0,-2x-1\rangle \cdot (\vec{r}_u \times \vec{r}_v)\, du\, dx$$

$$= \int_0^{2\pi}\int_0^2 \langle 0, 0, -2u\cos v - 1\rangle \cdot \langle 2u^2\cos v, -2u^2\sin v, u\rangle \, du\, dv$$

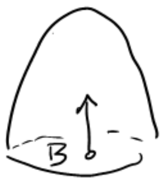
$$= \int_0^{2\pi}\int_0^2 (-2u^2\cos v - u)\, du\, dv = \int_0^{2\pi}\left[-\frac{2}{3}u^3\cos v - \frac{u^2}{2}\right]_0^2 dv = \int_0^{2\pi}\left(-\frac{16}{3}\cos v - 2\right)dv$$

$$= \left[-\frac{16}{3}\sin v - 2v\right]_0^{2\pi} = \left(-\frac{16}{3}\sin 2\pi - 2(2\pi)\right) - \left(-\frac{16}{3}\sin 0 - 2\cdot 0\right) = \boxed{-4\pi}$$

Notice we could also integrate over the circular base & get the same answer:

$$\iint_B \nabla \times F \cdot n \, d\sigma = \iint_B \langle 0,0,-2x-1\rangle \cdot \langle 0,0,1\rangle \, dA$$ switch to polar

$$= \iint_B (-2x-1)\, dA = \int_0^{2\pi}\int_0^2 (-2r\cos\theta - 1)\, r\, dr\, d\theta = \boxed{-4\pi}$$



Besides having many physical applications, Stokes theorem can also help evaluate a line- or surface-integral, if the other side is easier to do.