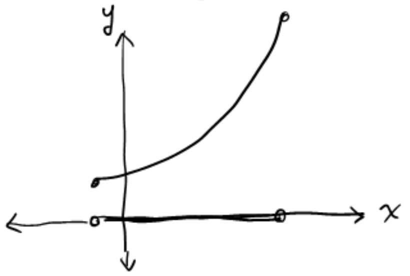
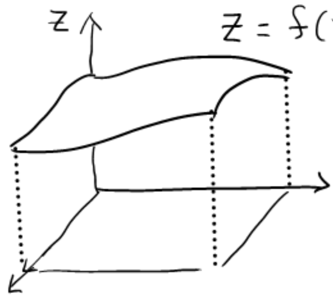
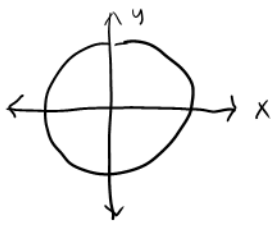
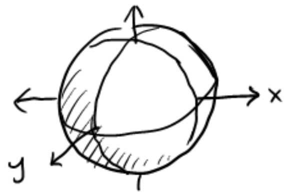
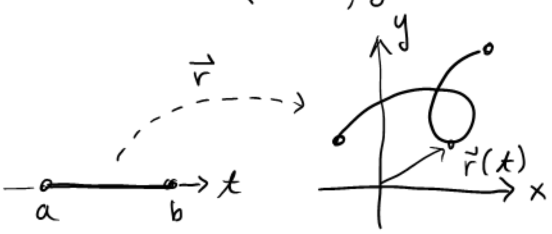
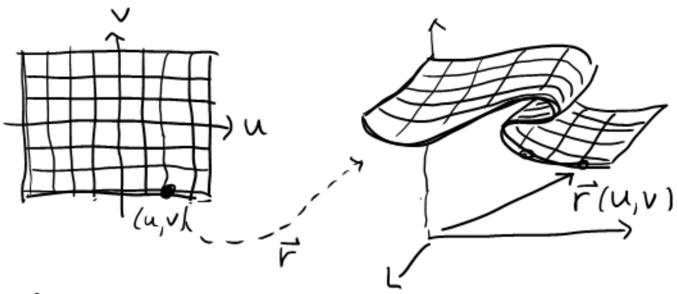
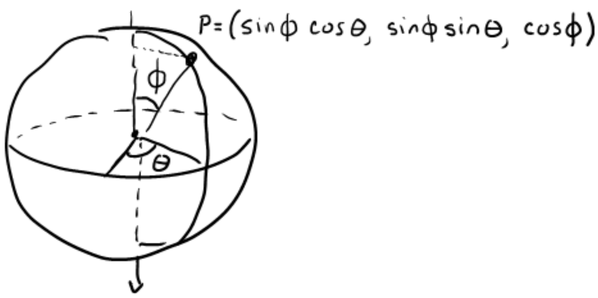
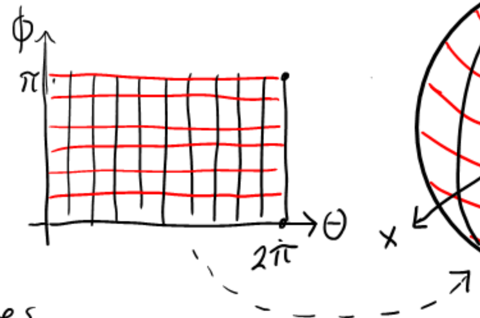
Section 16.5 Surface Area

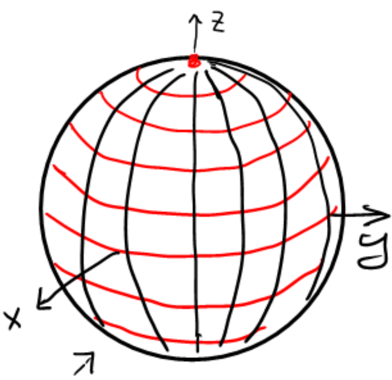
We have been studying line integrals. What's next? Surface integrals. In preparation for this we now examine surfaces and their areas. To begin, we lay out the ways of defining surfaces. We pair these with the ways of defining curves.

Ways of defining curves	Ways of defining surfaces
(A) Explicitly $y = f(x)$ 	(A) Explicitly $z = f(x,y)$ 
(B) Implicitly $f(x,y) = 0$ e.g. $x^2 + y^2 - 1 = 0$ 	(B) Implicitly $f(x,y,z) = 0$ e.g. $x^2 + y^2 + z^2 - 1 = 0$ 
(C) Parametrically $\vec{r}(t) = \langle f(t), g(t) \rangle$  $\vec{r}$ "wraps" interval $[a,b]$ along curve C	(C) Parametrically $\vec{r}(u,v) = \langle f(u,v), g(u,v), h(u,v) \rangle$  $\vec{r}$ "wraps" the rectangle on the uv-plane around the surface

Example Paremetric Description of unit sphere





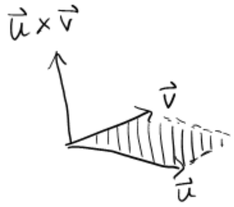


Any point P on sphere has spherical coordinates $(1, \phi, \theta)$ with $\rho = 1$, $0 \le \phi \le \pi$, $0 \le \theta \le 2\pi$. Cartesian Coordinates are $P(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$

$$\vec{r}(\theta,\phi) = \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi \rangle$$

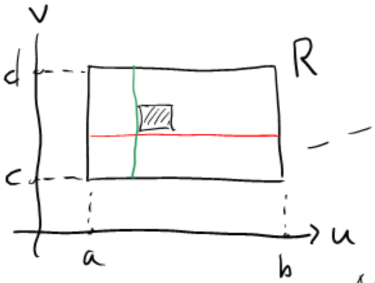
Surface Area

Recall this fact:
It's a key ingredient for computing surface area.

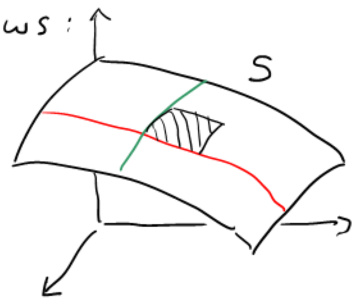


$|\vec{u} \times \vec{v}|$ = area of parallelogram spanned by $\vec{u}$ and $\vec{v}$.

Suppose a surface S is parametrized as follows:



$$\vec{r}(u,v) = \langle f(u,v),\ g(u,v),\ h(u,v) \rangle$$

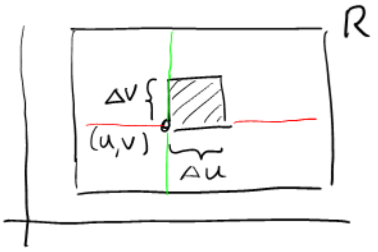


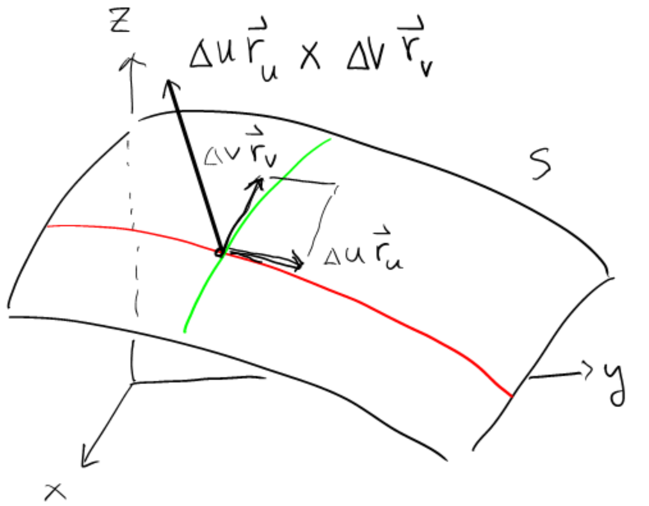
Note: Small rectangle on left maps to curved rectangle on S.

What is the area of S?

Let $\vec{r}_u = \left\langle \frac{\partial f}{\partial u}, \frac{\partial g}{\partial u}, \frac{\partial h}{\partial u} \right\rangle$

$\vec{r}_v = \left\langle \frac{\partial f}{\partial v}, \frac{\partial g}{\partial v}, \frac{\partial h}{\partial v} \right\rangle$





Area of curved rectangle $\approx |\Delta u \vec{r}_u \times \Delta v \vec{r}_v| = |\vec{r}_u \times \vec{r}_v| \Delta u \Delta v$

Now partition R into a grid of n small rectangles, where rectangle # k has dimensions Δu_k by Δv_k

$$\text{Area of } S \approx \sum_{k=1}^{n} |\vec{r}_u \times \vec{r}_v| \Delta u_k \Delta v_k$$

Now take $\lim\limits_{|P| \to 0}$ to get.

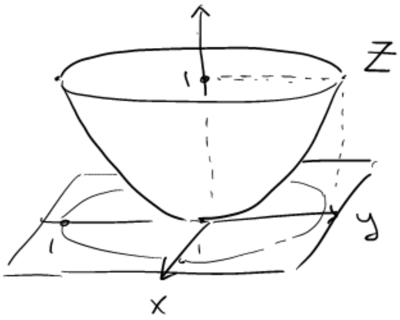
$$\text{Area of } S = \iint_R |\vec{r}_u \times \vec{r}_v|\, dA = \int_a^b \int_c^d |\vec{r}_u \times \vec{r}_v|\, dv\, du.$$

<u>Formula</u> Suppose a surface is parameterized as $\vec{r}(u,v)$ for $a \le u \le b$ and $c \le v \le d$. Then its area is

$$\iint_R |r_u \times r_v|\, dA = \int_a^b \int_c^d |\vec{r}_u \times \vec{r}_v|\, dv\, du$$

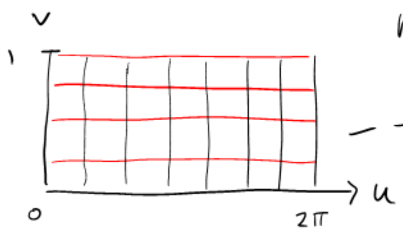
Example

Find the surface area of the paraboloid



$z = x^2 + y^2$ (over unit circle)

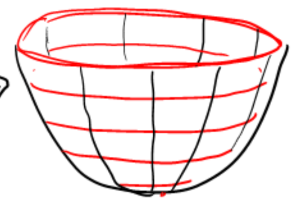
Parameterization:



$$r(u,v) = \langle v\cos u,\ v\sin u,\ v^2 \rangle$$

$$0 \le u \le 2\pi$$

$$0 \le v \le 1$$



$$\vec{r}_u = \langle -v\sin u,\ v\cos u,\ 0 \rangle$$

$$\vec{r}_v = \langle \cos u \quad \sin u \quad 2v \rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -v\sin u & v\cos u & 0 \\ \cos u & \sin u & 2v \end{vmatrix} = \langle 2v^2\cos u,\ 2v^2\sin u,\ -v \rangle$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{4v^4\cos^2 u + 4v^4\sin^2 u + v^2} = \sqrt{4v^4 + v^2} = v\sqrt{4v^2+1}$$

$$\text{Area} = \int_0^{2\pi}\int_0^1 |\vec{r}_u \times \vec{r}_v|\, dv\, du = \int_0^{2\pi}\int_0^1 \sqrt{4v^2+1}\ v\, dv\, du$$

$$w = 4v^2 + 1$$

$$dw = 8v\,dv$$

$$= \int_0^{2\pi}\int_0^1 \frac{1}{8}\sqrt{4v^2+1}\ 8v\, dv\, du = \int_0^{2\pi}\int_{4\cdot 0^2+1}^{4\cdot 1^2+1} \frac{1}{8}\sqrt{w}\, dw\, du$$

$$= \int_0^{2\pi} \left[\frac{1}{8}\frac{2}{3}\sqrt{w}^{\,3}\right]_1^5 du = \int_0^{2\pi} \frac{1}{12}\left(\sqrt{5}^{\,3} - \sqrt{1}^{\,3}\right) du$$

$$= \frac{5\sqrt{5}-1}{12}\int_0^{2\pi} du = 2\pi\,\frac{\sqrt{5}-1}{12} = \boxed{\frac{\pi}{6}(\sqrt{5}-1) \text{ square units}}$$

Note You can skip material on implicitly defined surfaces.

Advice: Work some exercises!