Section 16.3 Path Independence, Conservative Fields, Potential Functions (Continued)

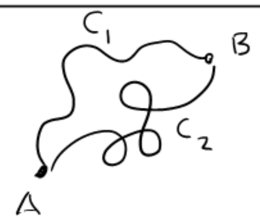
Recall the following:

Definition

Suppose for some vector field F it happens that

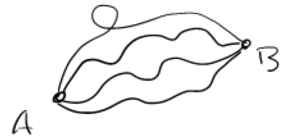
$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r}$$

whenever C and C_2 are two curves joining that that begin at the same point and end at the same point.



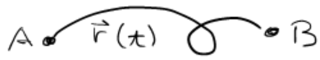
Then $\int_C \vec{F}\cdot d\vec{r}$ is said to be path independent.

A vector field $\vec{F}$ having this property is called a conservative v.f.



Note. For a conservative v.f. $\int_C \vec{F}\cdot d\vec{r}$ has the same value for all curves C joining A to B.

Theorem 1 Suppose $\vec{F} = \nabla f$ for some function $f(x,y,z)$ (or $f(x,y)$). Then for any curve C



in the domain of f,

$$\int_C \vec{F}\cdot d\vec{r} = f(B) - f(A)$$

i.e. $$\int_C \nabla f\cdot d\vec{r} = f(B) - f(A)$$

Proof Suppose $\vec{F} = \nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$ for some function $f(x,y,z)$ and C is $\vec{r}(t)$, $a \le t \le b$.

$$\int_C F\cdot dr = \int_C F\cdot \frac{dr}{dt}\, dt$$

$$= \int_a^b \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle dt = \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \right) dt$$

$$= \int_a^b \frac{d}{dt}\left[f(g(t), h(t), k(t)) \right] dt \qquad \text{(chain Rule)}$$

$$= f(g(b), h(b), k(b)) - f(g(a), h(b), k(b)) \qquad \text{(Fundamental Theo. of Calc.)}$$

$$= f(B) - f(A)$$

Theorem 2 $\left(\vec{F} = \nabla f \text{ for some function } f \right) \Longleftrightarrow \left(\vec{F} \text{ is conservative} \right)$

Theorem 3 $(F \text{ conservative}) \Longleftrightarrow \left(\int_C F\cdot dr = 0 \text{ for any closed curve } C \right)$

Definition If $\vec{F} = \nabla f$, then f is called a potential function for

Example: $\vec{F}(x,y,z) = \langle zy\cos(xy),\ zx\cos(xy),\ \sin(xy) \rangle$

Potential function for $\vec{F}$ is $f(x,y,z) = z\sin(xy)$ because $\vec{F} = \nabla f$

Note: Another potential function is $f(x,y,z) = z\sin(xy) + 5$

Compare:

F.T.C $\int_a^b F(t)\,dt = f(b) - f(a)$ where $f'(t) = F(t)$ (f is antiderivative of F

Theorem 1 $\int_C \vec{F} \cdot d\vec{r} = f(B) - f(A)$ where $\nabla f = F$ (f is potential function of F

Therefore potential functions are "antiderivatives" of vector fields.

Questions

① How can we tell if $\vec{F}$ is consertave?

② If $\vec{F}$ is consertave, how can we find its potential function?

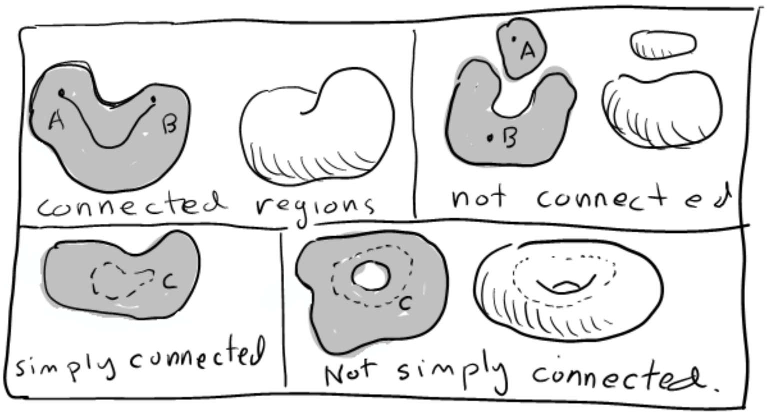
③ How can all this be useful?

To answer these questions fully we will need to make some assumptions concerning regions on which functions are defined.

A region R is connected if any two points A & B in R can be joined by a curve that lies entirely in R

A region is simply connected if any closed curve in R can be shrunk to a point without leaving D.

This section's theorems often assume F is defined on simply connected regions containing the curve in question.



Question 1 How do we know if F is conservative?

First, suppose it is. Then $F = \nabla f = \langle M, N, P \rangle = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$ for some function $f(x,y,z)$. Note that we must then have

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial N}{\partial y} \qquad \text{i.e } \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\frac{\partial N}{\partial z} = \frac{\partial}{\partial z}\frac{\partial f}{\partial y} = \cdots\cdots = \frac{\partial}{\partial y}\frac{\partial f}{\partial z} = \frac{\partial P}{\partial y} \qquad \text{i.e. } \frac{\partial N}{\partial z} = \frac{\partial P}{\partial y}$$

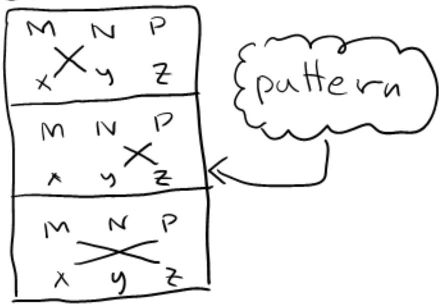
$$\frac{\partial P}{\partial x} = \frac{\partial}{\partial x}\frac{\partial f}{\partial z} = \cdots\cdots = \frac{\partial}{\partial z}\frac{\partial f}{\partial x} = \frac{\partial M}{\partial z} \qquad \text{i.e. } \frac{\partial P}{\partial x} = \frac{\partial M}{\partial z}$$

Thus These three conditions must hold if $F = \langle M, N, P \rangle$ is conservative.

Component Test For Conservative Fields

Suppose $\vec{F} = \langle M, N, P \rangle$ and has a simply connected domain. Then:

$$\left(\vec{F} \text{ is conservative}\right) \iff \begin{cases} \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x} \\ \dfrac{\partial N}{\partial z} = \dfrac{\partial P}{\partial y} \\ \dfrac{\partial P}{\partial x} = \dfrac{\partial M}{\partial z} \end{cases}$$



Also:

$$\left(\vec{F} = \langle M, N \rangle \text{ conservative}\right) \iff \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Example:

Conservative or not? $\vec{F}(x, y, z) = \langle \underbrace{zy\cos(xy)}_{M}, \underbrace{zx\cos(xy)}_{N}, \underbrace{\sin(xy)}_{P} \rangle$

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= z\cos(xy) - xyz\sin(xy) = \frac{\partial N}{\partial x} \\ \frac{\partial N}{\partial z} &= x\cos(xy) = \frac{\partial P}{\partial y} \\ \frac{\partial P}{\partial x} &= y\cos(xy) = \frac{\partial M}{\partial z} \end{aligned}\right\}$$

From this, test shows that F is indeed conservative.

Question 2 Once we know a field is conservative, how do we find a potential function for it?

For example, find a potential function for

$$\vec{F}(x, y, z) = \langle zy\cos(xy), zx\cos(xy), \sin(xy) \rangle = \overbrace{\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle}^{\text{seek this.}}$$

$$\frac{\partial f}{\partial z} = \sin(xy) \rightsquigarrow f(x,y,z) = \int \sin(xy)\, dz = z\sin(xy) + \underbrace{g(x,y)}_{\text{constant } C \text{ as far as } z \text{ is concerned}}$$

Thus $f(x,y,z) = z\sin(xy) + g(x,y) \quad (*)$

We will know f as soon as we determine g.

Note $$\frac{\partial f}{\partial x} = \underbrace{zy\cos(xy)}_{\text{First component of } F} = \underbrace{z\cos(xy) + \frac{\partial}{\partial x}\left[g(x,y)\right]}_{\frac{\partial}{\partial x} \text{ of } (*), \text{ above}}$$

From above equation, get $\frac{\partial}{\partial x}[g(x,y)] = 0$, so $g(x,y)$ is a constant as far as x is concerned. Thus $g(x,y) = h(y)$, so $f(x,y,z) = z\sin(xy) + h(y)$.

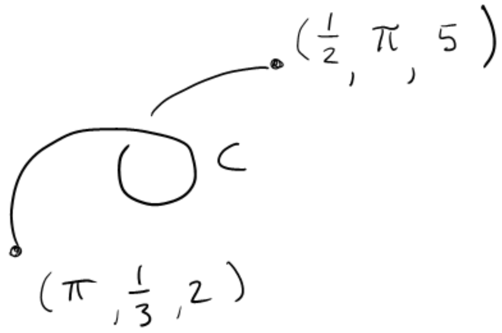
Now $\frac{\partial f}{\partial y} = zx\cos(xy) = zx\cos(xy) + h'(y) \Rightarrow h'(y) = 0 \Rightarrow h(y) = C$

ANSWER Potential Function is $\boxed{f(x,y,z) = z\sin(xy) + C}$

Question 3 How can this be useful?

Well, it simplifys computations of some line integrals.

Example



Let C be a curve $\vec{r}(t)$ joining $(\pi, \frac{1}{3}, 2)$ to $(\frac{1}{2}\pi, 5)$ and let

$$\vec{F} = \langle zy\cos(xy),\ zx\cos(xy),\ \sin(xy) \rangle$$

Compute $\displaystyle\int_C \vec{F} \cdot d\vec{r}$

Solution Because we know $\vec{F} = \nabla f$, where where $f(x, y, z) = z\sin(xy)$, Theorem 1 says

$$\int_C \vec{F} \cdot d\vec{r} = f(\tfrac{1}{2}, \pi, 5) - f(\pi, \tfrac{1}{3}, 2)$$

$$= 5\sin\left(\tfrac{1}{2}\pi\right) - 2\sin\left(\pi\tfrac{1}{3}\right)$$

$$= 5 \cdot 1 - 2\frac{\sqrt{3}}{2} = \boxed{5 - \sqrt{3}}$$

Read material on <u>exact</u> <u>differentials</u>

Work some exercises!