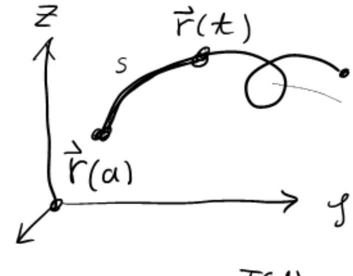
Section 16.2 Line Integrals over Vector Fields (Continued)

- For today C designates a curve $\vec{r}(t) = \langle g(t), h(t), k(t) \rangle$, $a \le t \le b$.

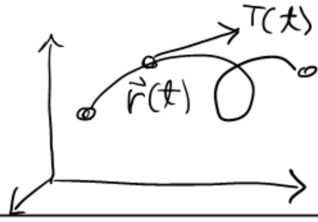
Thus $\dfrac{d\vec{r}}{dt} = \vec{v}(t) = \langle g'(t), h'(t), k'(t) \rangle = \left\langle \dfrac{dx}{dt}, \dfrac{dy}{dt}, \dfrac{dz}{dt} \right\rangle$

so $\boxed{d\vec{r} = \vec{v}(t)\, dt}$



- Also arc length from $\vec{r}(a)$ to $\vec{r}(t)$ is $s = \int_a^t |\vec{v}(u)|\, du$

so $\dfrac{ds}{dt} = |\vec{v}(t)|$ and $\boxed{ds = |\vec{v}(t)|\, dt}$



- Unit tangent vector at $\vec{r}(t)$ is $T(t) = \dfrac{\vec{v}(t)}{|\vec{v}(t)|} = \dfrac{\vec{v}(t)\, dt}{|\vec{v}(t)|\, dt} = \dfrac{d\vec{r}}{ds}$

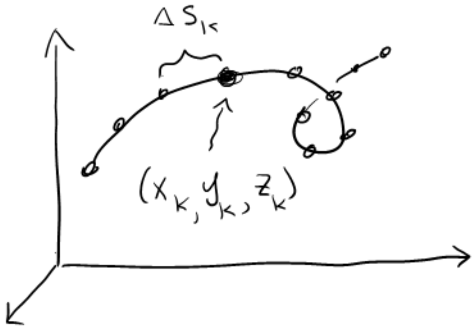
Note: $T(t)\, ds = \dfrac{d\vec{r}}{ds} ds = d\vec{r} = \vec{v}(t)\, dt$

From §16.1

The line integral of $f(x,y,z)$ along C is

$$\int_C f\, ds = \lim_{\|P\| \to 0} \sum_{k=1}^{n} f(x_k, y_k, z_k) \Delta s_k$$

$$\int_C f\, ds = \int_a^b f(g(t), h(t), k(t))\, |v(t)|\, dt$$

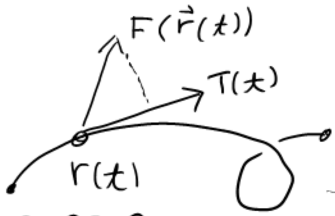


From §16.2 Line integral of a vector field.

Consider vector field $\vec{F}(x,y,z) = \langle M(x,y,z), N(x,y,z), P(x,y,z) \rangle = \langle M, N, P \rangle$.

Get scalar function $f(t) = \vec{F}(\vec{r}(t)) \cdot T(t) = \vec{F} \cdot T$

The line integral of $\vec{F}$ along C is



$$\int_C \vec{F}(\vec{r}(t)) \cdot T(t)\, ds = \int_C F \cdot T\, ds$$

$= \int f\, ds$. Just a regular line integral

Thus $\displaystyle\int_C F \cdot T\, ds = \int_a^b F(g(t), h(t), k(t)) \cdot \vec{v}(t)\, dt$

Just a regular integral you can evaluate

Ways of writing it:

$$\int_C \vec{F} \cdot T\, ds = \int_C \vec{F} \cdot \frac{d\vec{r}}{ds} ds = \underbrace{\int_C \vec{F} \cdot d\vec{r}}_{\text{most common}} = \int_a^b F \cdot \frac{d\vec{r}}{dt} dt = \int_a^b \left(M \frac{dx}{dt} + N \frac{dy}{dt} + P \frac{dz}{dt} \right) dt$$

$$= \int_a^b M\, dx + N\, dy + P\, dz$$

$$= \int_a^b M\, dx + \int_a^b N\, dy + \int_a^b P\, dz$$

But what to make of $\int_a^b M\,dx$?

Since a & b are t values, the integration must be with respect to t.

$$\int_a^b M\,dx = \int_a^b M \frac{dx}{dt}\,dt = \int_a^b M(g(t), h(t), k(t))\, g'(t)\,dt$$

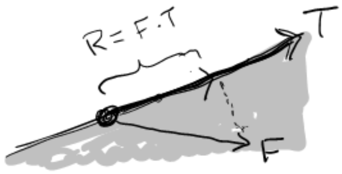
Just a regular integral

$$\int_a^b N\,dy = \text{- - - - - -} \quad \int_a^b N(g(t), h(t), k(t))\, h'(t)\,dt$$

$$\int_a^b P\,dz = \text{- - - - - - - -} \quad \int_a^b P(g(t), h(t), k(t))\, k'(t)\,dt$$

Interpretations of Line Integrals of Vector Fields

WORK Line integrals of vector fields can be used to compute work.

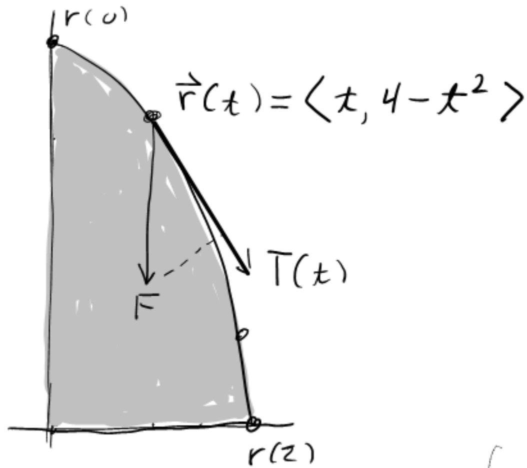


Recall If a force F acts on an object confined to a path with (unit) direction T, then the resultant force on the object is R = F · T

Recall If force of F·T pounds moves object a distance S, then the work done is W = (Force)(Distance) = F·T S foot pounds

Example

Suppose gravity moves an object down the curve $\vec{r}(t) = \langle t, 4-t^2 \rangle$, $0 \le t \le 2$



How much work is done?

Note The force of gravity is a (constant) vector field $F(x,y) = \langle 0, -w \rangle$ where w is the weight of the object,

Approximate curve by n straight line segments of lengths $\Delta s_1, \Delta s_2, \dots, \Delta s_k, \dots, \Delta s_n$. Then $W \approx \sum_{k=1}^{n} F \cdot T\, \Delta s_k$

$$W = \lim_{|P|\to 0} \sum_{k=1}^{n} F \cdot T\, \Delta s_k = \int_C F \cdot T\, ds = \int_0^2 F \cdot \frac{d\vec{r}}{dt}\,dt$$

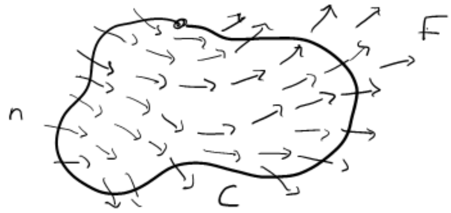
$$= \int_0^2 \langle 0, -w \rangle \cdot \langle 1, -2t \rangle\,dt = \int_0^2 2tw\,dt = \left[t^2 w \right]_0^2$$

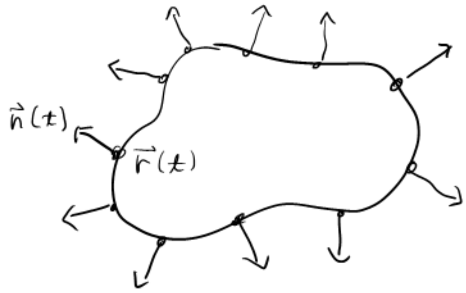
$$= 4w \text{ foot pounds.}$$

[Read further examples in text.]

FLUX CALCULATIONS

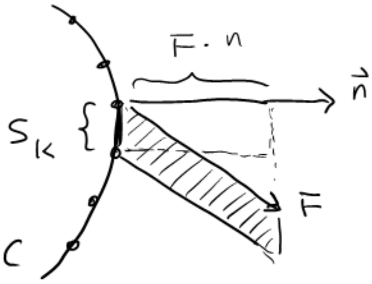
Suppose C is a closed curve (i.e. beginning and ending at the same point) enclosing a region in the plane. Also there is a vector field $F(x,y) = \langle M(x,y), N(x,y) \rangle = \langle M, N \rangle$ representing the velocity of a fluid flowing in the plane. The flux is the net flow into (or out of) the region.





How do you compute flux?

Let $\vec{n}(t)$ be the outward pointing normal vector to the curve at $\vec{r}(t)$.



Divide C into segments $\Delta S_1\ \Delta S_2 \cdots \Delta S_K$

Net flow over ΔS_K is area of shaded region

$= (\text{height})(\text{base}) = F \cdot \vec{n}\, \Delta S_K$

Thus flux $\approx \sum_{K=1}^{n} F \cdot \vec{n}\, \Delta S_K$

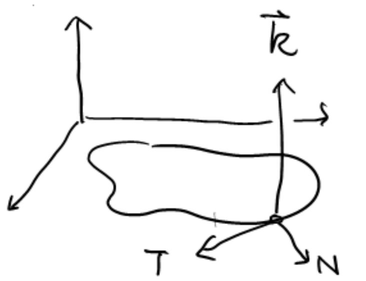
$$\text{Thus } \boxed{\text{Flux} = \lim_{|P| \to 0} \sum_{K=1}^{n} F \cdot \vec{n}\, \Delta S_K = \int_C F \cdot \vec{n}\, ds}$$

Computing the flux integral thus involves finding $N(t)$

Note: $\vec{n}(t) = T(t) \times \vec{k}$

$$= \begin{vmatrix} i & j & k \\ \frac{dx}{ds} & \frac{dy}{ds} & 0 \\ 0 & 0 & 1 \end{vmatrix} =$$

$$= \left\langle \frac{dy}{ds}, -\frac{dx}{ds}, 0 \right\rangle \rightsquigarrow \left\langle \frac{dy}{ds}, -\frac{dx}{ds} \right\rangle$$



$$T(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, 0 \right\rangle \frac{1}{|v(t)|}$$

$$= \left\langle \frac{dx}{dt}, \frac{dy}{dt}, 0 \right\rangle \frac{dt}{|v(t)|\, dt}$$

$$= \langle dx, dy, 0 \rangle \frac{1}{ds}$$

$$= \left\langle \frac{dx}{ds}, \frac{dy}{ds}, 0 \right\rangle$$

$$\text{Therefore Flux} = \int_C F \cdot \vec{n}\, ds$$

$$= \int_C \langle M, N \rangle \cdot \left\langle \frac{dy}{ds}, -\frac{dx}{ds} \right\rangle ds$$

$$= \int_C \left(M \frac{dy}{ds} - N \frac{dx}{ds} \right) ds$$

See examples in text

$$= \int_C M\,dy - N\,dx$$

$$= \int_a^b M(g(t), h(t))\, h'(t)\, dt - \int_a^b N(g(t), h(t))\, g'(t)\, dt$$