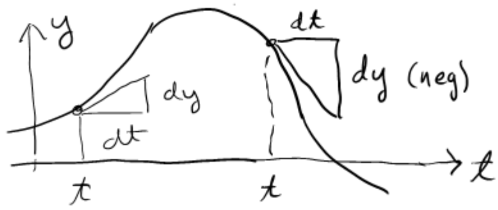
Section 16.2 Vector Fields and Line Integrals

Before getting started, we review the topic of differentials, because they will come up a lot and understanding them will make our work easier.

If $y = f(t)$, the differentials dy and dt are variables related by the equation $dy = f'(t)\,dt$, so $\frac{dy}{dt} = f'(t)$.



At any t, $f'(t)$ has a particular value, and at point t differentials dy and dt are related by the linear equation $dy = f'(t)\,dt$. This means that at t, incrementing t by dt has the effect of changing $y = f(t)$ by $dy = f'(t)\,dt$.

- If $\vec{r}(t) = \langle g(t), h(t), k(t) \rangle$ then $\frac{d\vec{r}}{dt} = \vec{v}(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle$ so

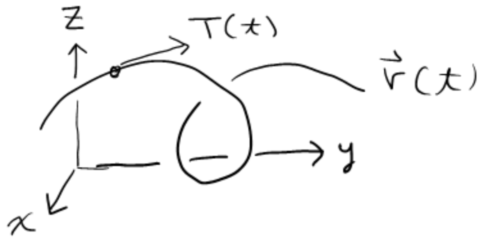
$$\boxed{d\vec{r} = \vec{v}(t)\,dt} = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle dt = \langle dx, dy, dz \rangle$$

- Arc length from $\vec{r}(a)$ to $r(t)$ is $s = \int_a^t |\vec{v}(u)|\,du$

so $\frac{ds}{dt} = \frac{d}{dt}\left[\int_a^t |\vec{v}(u)|\,du \right] = |\vec{v}(t)|$. $\Rightarrow \boxed{ds = |\vec{v}(t)|\,dt}$

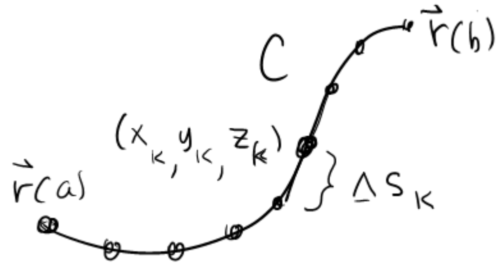
- The unit tangent vector to $\vec{r}(t)$ at t is

$$T(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|} = \frac{\vec{v}(t)\,dt}{|\vec{v}(t)|\,dt} = \frac{d\vec{r}}{ds}$$



- Note: $d\vec{r} = \frac{d\vec{r}}{ds}\,ds = \frac{d\vec{r}}{dt}\,dt$

Recall: The line integral of $f(x,y,z)$ over curve C: $\vec{r}(t) = \langle g(t), h(t), k(t) \rangle$ $a \le t \le b$ is



$$\int_C f(x,y,z)\,ds = \lim_{|P| \to 0} \sum_{k=1}^{n} f(x_k, y_k, z_k)\,\Delta S_k$$

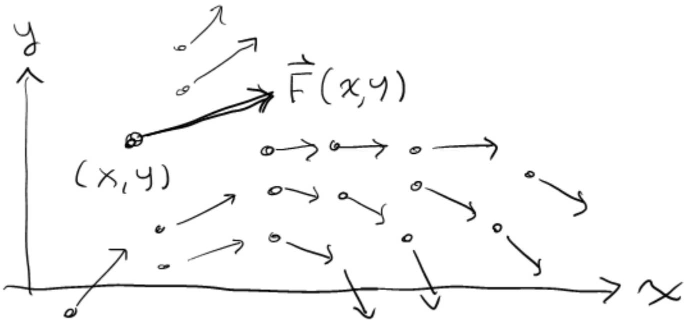
$$= \int_a^b f(g(t), h(t), k(t))\,|\vec{v}(t)|\,dt$$

Now that we've reviewed differentials and line integrals, it's time to introduce the next ingredient: Vector Fields.

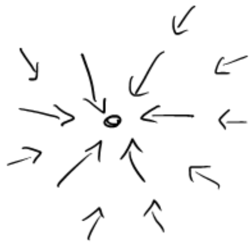
Soon we will weave all this together.

Vector Fields

A vector field in the plane is a function $\vec{F}(x,y) = \langle M(x,y), N(x,y) \rangle$ that assigns a vector to each point (x,y) in its domain



A vector field in 3-D space is a function $\vec{F}(x,y,z) = \langle M(x,y,z), N(x,y,z), P(x,y,z) \rangle$ assigning a vector to each point (x,y,z).



Example: gravitational force field.

Example Gradient field: Given $f(x,y,z)$

$$\vec{F}(x,y,z) = \nabla f(x,y,z) = \left\langle \underbrace{\frac{\partial f}{\partial x}}_{M(x,y,z)}, \underbrace{\frac{\partial f}{\partial y}}_{N(x,y,z)}, \underbrace{\frac{\partial f}{\partial z}}_{P(x,y,z)} \right\rangle$$

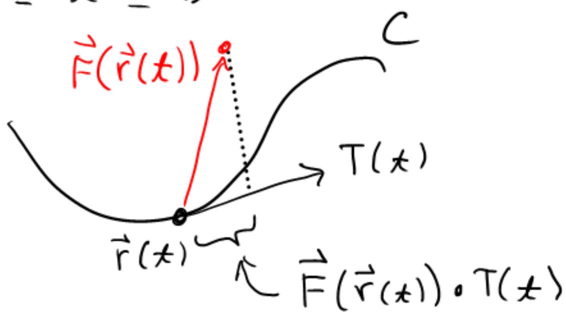
Abbreviation $\vec{F} = \langle M, N, P \rangle$

Line Integrals of Vector Fields

For the moment, lets work with 2-D vector fields becaus the pictures are easier. Once we get a handle on this it's easy to adapt it to 3-D.

Consider vector field $\vec{F}(x,y)$ on plane and curve C: $\vec{r}(t) = \langle g(t), h(t) \rangle$, $a \le t \le b$.

At each point $\vec{r}(t)$ on C there is a unit tangent $\vec{T}(t) = \dfrac{\vec{v}(t)}{|v(t)|}$ and a vector $\vec{F}(\vec{r}(t))$ from the vector field.



Get scalar function of t:

$$\vec{F}(\vec{r}(t)) \cdot T(t) = \vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{ds}$$ defined at any $\vec{r}(t)$ on curve

Line integral of $\vec{F}$ along C is

$$\int_C F(\vec{r}(t)) \cdot T(t)\, ds = \int_C F \cdot T\, ds = \int_C F \cdot \frac{d\vec{r}}{ds}\, ds = \int_C F \cdot d\vec{r}$$

How to compute it

$$\int_C \vec{F}(\vec{r}(t)) \cdot T(t)\, ds = \int_a^b \vec{F}(r(t)) \cdot \overbrace{\frac{v(t)}{|v(t)|}}^{T(t)} \overbrace{|v(t)|\, dt}^{ds}$$

$$= \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{v}(t)\, dt$$

$$= \int_a^b \vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}\, dt$$

<u>Example</u> $\vec{F}(x, y, z) = \langle z, xy, -y^2 \rangle$

$C: \vec{r}(t) = \langle t^2, t, \sqrt{t} \rangle, \quad 0 \le t \le 1$

Compute line integral of $\vec{F}$ along C.

<u>Solution</u> Formula says we need to compute $\vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}$ and then integrate this.

$$\vec{F}(r(t)) = \langle \sqrt{t}, t^2 t, -t^2 \rangle = \langle \sqrt{t}, t^3, -t^2 \rangle$$

$$\frac{d\vec{r}}{dt} = \langle 2t, 1, \frac{1}{2\sqrt{t}} \rangle$$

$$\vec{F}(\vec{r}(t)) \cdot \frac{dr}{dt} = \sqrt{t}\, 2t + t^3 \cdot 1 - t^2 \frac{1}{2\sqrt{t}} = 2t^{\frac{3}{2}} + t^3 - \frac{1}{2} t^{3/2}$$

$$= \frac{3}{2} t^{3/2} + t^3$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \frac{dr}{dt}\, dt$$

$$= \int_a^b \frac{3}{2} t^{\frac{3}{2}} + t^3\, dt$$

$$= \left[\frac{3}{2} \frac{2}{5} t^{\frac{5}{2}} + \frac{t^4}{4} \right]_0^1 = \left[\frac{3}{5} \sqrt{t}^{\,5} + \frac{1}{4} t^4 \right]_0^1$$

$$= \frac{3}{5} + \frac{1}{4} = \frac{12}{20} + \frac{5}{20} = \boxed{\frac{17}{20}}$$