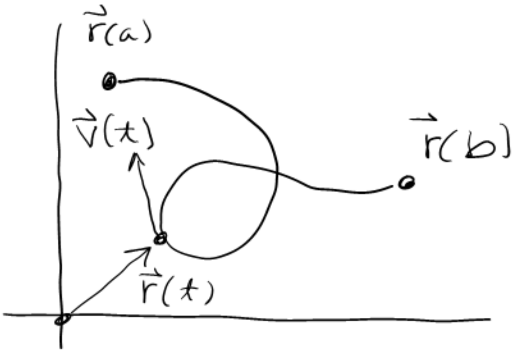
Chapter 16 Integration in Vector Fields.

Curves in space play a big role in this chapter, so let's review them now

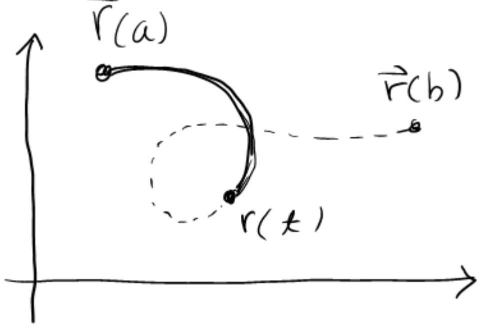


Curve in plane $\vec{r}(t) = \langle g(t), h(t) \rangle$, $a \le t \le b$

Velocity vector $\vec{v}(t) = \vec{r}\,'(t) = \langle g'(t), h'(t) \rangle$

Speed at time t: $|\vec{v}(t)| = \sqrt{(g'(t))^2 + (h'(t))^2}$

Arc length : $\int_a^b |v(t)|\, dt$



Arc length from $t = a$ to t is

$$s(t) = \int_a^t |\vec{v}(u)|\, du$$

Then $s'(t) = \frac{d}{dt}\left[\int_a^t |\vec{v}(u)|\, du \right] = |\vec{V}(t)|$

That is, $\frac{ds}{dt} = |\vec{v}(t)|$.

Consequently $\boxed{ds = |\vec{v}(t)|\, dt}$ ← $\underbrace{\text{distance}}_{ds} = \underbrace{\text{speed}}_{|v(t)|} \cdot \underbrace{\text{time}}_{dt}$

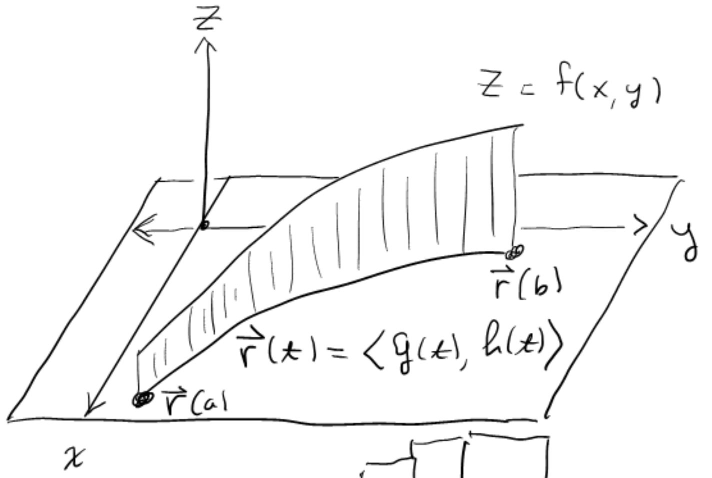
This means that at time t, a small change dt in t effects a change $ds = |\vec{v}(t)|\, dt$ in s.

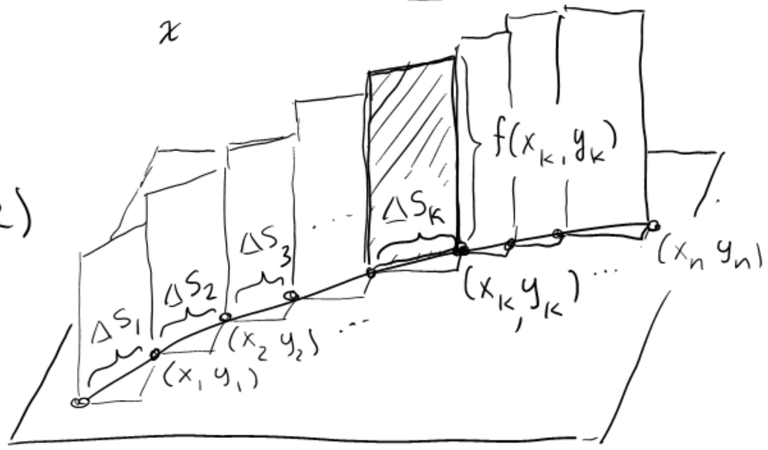
Everything that was noted above also holds for curves in space.

Section 16.1 Line Integrals

Motivational Question

What is the area of the curved region above the curve $\vec{r}(t) = \langle g(t), h(t) \rangle$, $a \le t \le b$ and below graph of $z = f(x,y)$?





Answer:

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} (\text{Area of rectangle } \# k)$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k)\, \Delta S_k$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} f(g(t_k), h(t_k)) \sqrt{\Delta x_k^2 + \Delta y_k^2}$$

$$= \lim_{n\to\infty} \sum_{k=1} f(g(t_k), h(t_k)) \sqrt{\left(\frac{\Delta x_k}{\Delta t_k}\right)^2 + \left(\frac{\Delta y_k}{\Delta t_k}\right)^2}\ \Delta t_k$$

$$= \int_a^b f(g(t), h(t)) \sqrt{g'(t)^2 + h'(t)^2}\, dt$$

$$= \int_a^b f(g(t), h(t))\, |v(t)|\, dt$$

$$= \int_a^b f(g(t), h(t))\, ds$$

$a \quad t_1\, t_2 \cdots t_k \cdots b \qquad t$

$(x_k, y_k) = (g(t_k), h(t_k))$

$\Delta x_k = g(t_k) - g(t_{k+1})$

$\Delta y_k = h(t_k) - h(t_{k+1})$

Definitions

Let C be the plane curve $r(t) = \langle g(t), h(t) \rangle$ for $a \le t \le b$

The line integral of $f(x,y)$ over C is

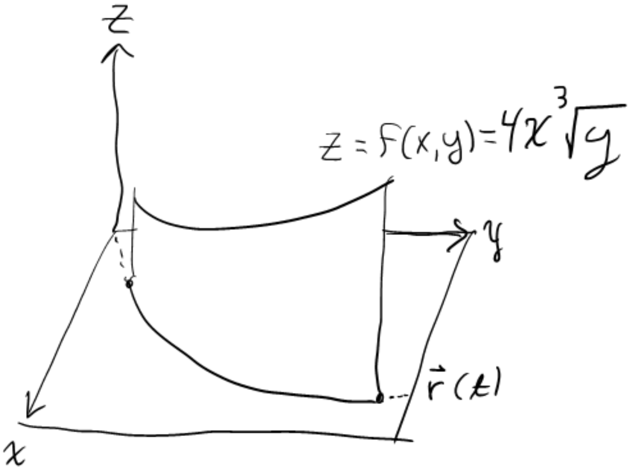
$$\int_C f(x,y)\, ds = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k)\, \Delta S_k = \int_a^b f(g(t), h(t))\, |v(t)|\, dt$$

If C is $r(t) = \langle g(t), h(t), k(t) \rangle$ for $a \le t \le b$ (in 3-D space)

then the line integral of $f(x,y,z)$ over C is

$$\int_C f(x,y,z)\, ds = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k, z_k)\, \Delta S_k = \int_a^b f(g(t), h(t), k(t))\, |v(t)|\, dt$$

Example C is curve $\vec{r}(t) = \langle t, t^4 \rangle$ for $1 \le t \le 2$. Find area of curved region above C, and below $z = f(x,y) = 4x^3\sqrt{y}$



Answer $\int_C f(x,y)\,ds$

$$= \int_1^2 f(t, t^4)\,|v(t)|\,dt$$

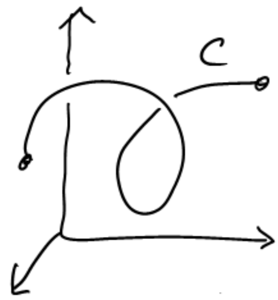
$$= \int_1^2 4t^3\sqrt{t^4}\,\sqrt{1^2 + (4t^3)^2}\,dt = \int_1^2 \sqrt{1+16t^6}\,4t^5\,dt$$

$$= \frac{1}{24}\int_1^2 \sqrt{1+16t^6}\,96t^5\,dt = \frac{1}{24}\int_{1+16\cdot 1^6}^{1+16\cdot 2^6} \sqrt{u}\,du \qquad \left\{\begin{array}{l} u = 1+16t^6 \\ du = 96t^5\,dt \end{array}\right\}$$

$$= \frac{1}{24}\left[\frac{2\sqrt{u}^3}{3}\right]_{17}^{1025} = \frac{1}{36}\left(1025\sqrt{1025} - 17\sqrt{17}\right) \approx \boxed{909.6088 \text{ sq units.}}$$

Note The square root in $|v(t)|$ can make these integrals hard to evaluate by standard methods, sometimes.

Other Interpretations



Suppose $\vec{r}(t) = \langle g(t), h(t), k(t) \rangle$, $a \le t \le b$ describes a wire in space, and its density at (x,y,z) is $\delta(x,y,z)$.

Then: Mass of wire $= M = \int_C \delta(x,y,z)\,ds$

$$\text{Moments}\left\{\begin{array}{l} M_{yz} = \int_C x\,\delta(x,y,z)\,ds \\ M_{xz} = \int_C y\,\delta(x,y,z)\,ds \\ M_{xy} = \int_C z\,\delta(x,y,z)\,ds \end{array}\right.$$

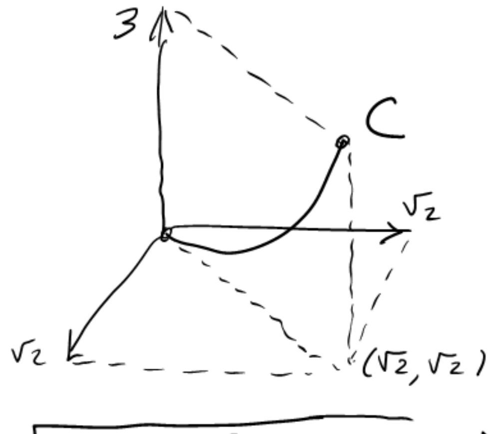
Center of mass $(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_{yz}}{M}, \frac{M_{xz}}{M}, \frac{M_{xy}}{M}\right)$

NOTE You can ignore material on Moments of inertia

Example § 16.1 (35)

Find the mass of the wire $\vec{r}(t) = \langle \sqrt{2}\,t,\ \sqrt{2}\,t,\ 4-t^2 \rangle$, $0 \le t \le 1$ if density is $\delta(t) = 3t$.

Here they mean density at point $\vec{r}(t)$



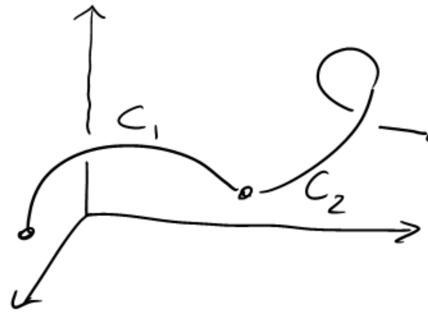
$$M = \int_C \delta(t)\,ds = \int_C 3t\,ds = \int_0^1 3t\sqrt{\sqrt{2}^2+\sqrt{2}^2+(-2t)^2}\,dt$$

$$= \int_0^1 \sqrt{4+4t^2}\,3t\,dt = 6\int_0^1 \sqrt{1+t^2}\,t\,dt = 3\int_0^1 \sqrt{1+t^2}\,2t\,dt$$

$$= 3\int_{1+0^2}^{1+1^2} \sqrt{u}\,du = \left[2\sqrt{u}^3\right]_1^2 = \boxed{4\sqrt{2} - 2 \text{ grams}}$$

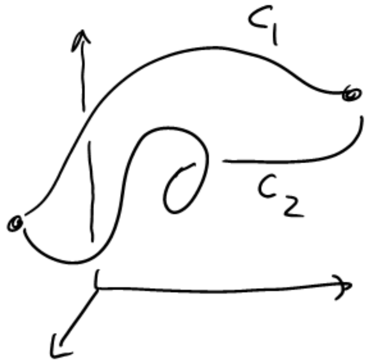
Additivity of line integrals:

$$\int_{C_1 \cup C_2} f(x,y,z)\,ds = \int_{C_1} f(x,y,z)\,ds + \int_{C_2} f(x,y,z)\,ds$$



Important Note

Given two path C_1 and C_2 that have common starting and ending points, it is typpically the case that



$$\int_{C_1} f\,ds \neq \int_{C_2} f\,ds$$

Later we will investigate situations in which equality holds.