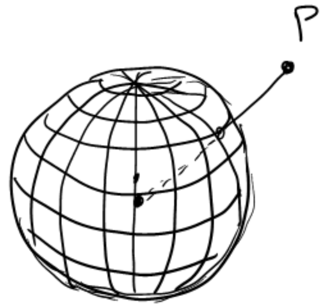
Section 15.7 (Continued) Spherical Coordinates

For some computations, spherical coordinates are the best choice, or tool. Here's how they work.

Basic Idea

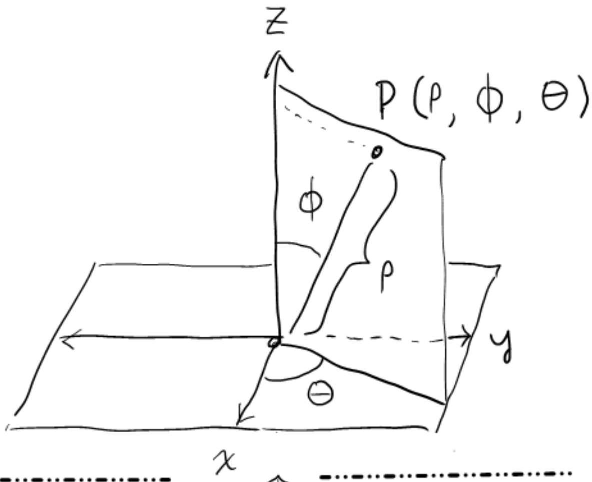
Any point P above or below Earth's surface can be described by three numbers:

ϕ = Latitude
θ = Longitude
ρ = Distace from center of Earth

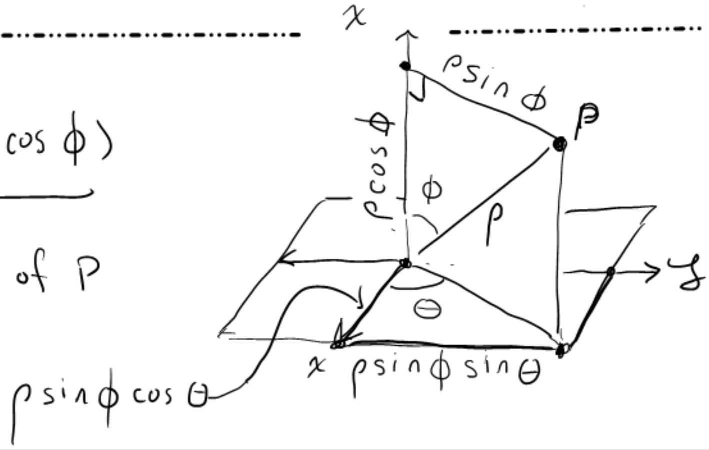


Spherical Coordinates

Any point P in space can be located by its <u>spherical coordinates</u> (ρ, ϕ, θ) where ρ is its distance from the origin, ϕ is its angle of inclination to the z-axis and θ is its polar angle.

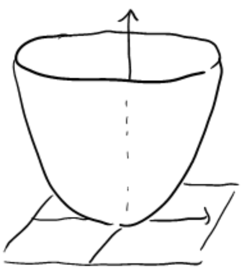


$$\underbrace{(\rho, \phi, \theta)}_{\text{spherical coordinates of } P} \rightsquigarrow \underbrace{(\rho \sin\phi \cos\theta,\ \rho \sin\phi \sin\theta,\ \rho \cos\phi)}_{\text{Cartesian coordinates of } P}$$



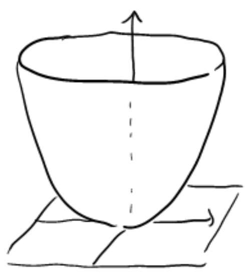
Surfaces can be defined by equations involving either Cartesian, polar or spherical coordinates.

Cartesian



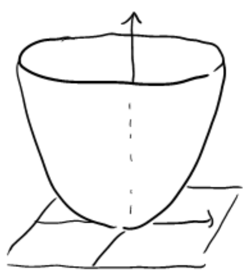
$$\boxed{z = x^2 + y^2}$$

Cylindrical



$$z = x^2 + y^2$$
$$z = (r\cos\theta)^2 + (r\sin\theta)^2$$
$$\boxed{z = r^2}$$

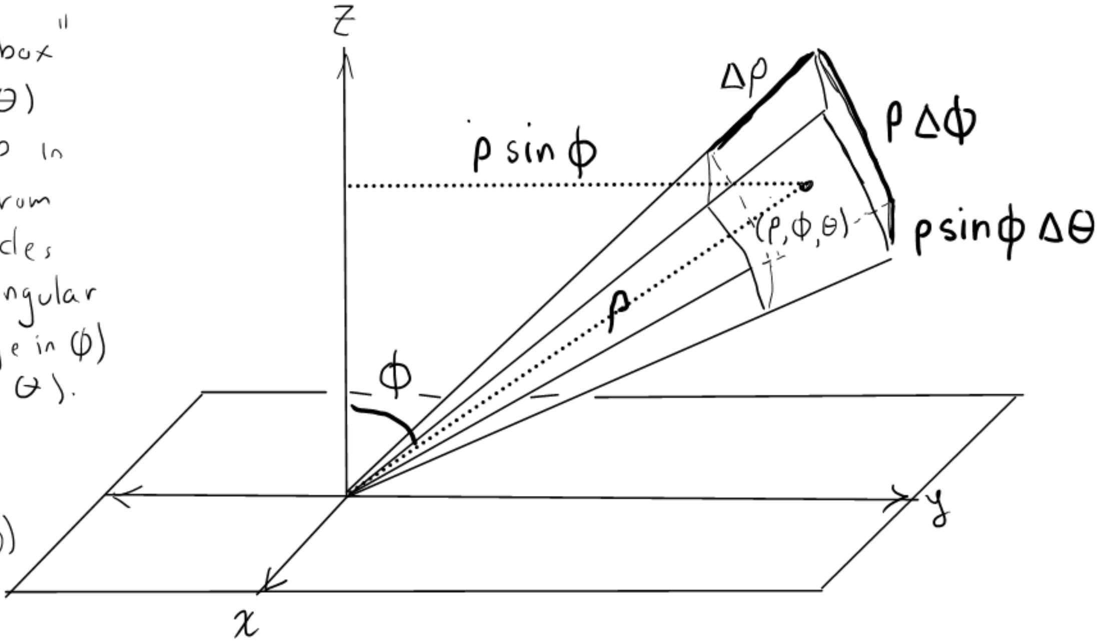
Spherical



$$z = x^2 + y^2$$
$$\rho\cos\phi = (\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2$$
$$\rho\cos\phi = \rho^2\sin^2\phi$$
$$\boxed{\rho = \cos\phi\csc^2\phi}$$

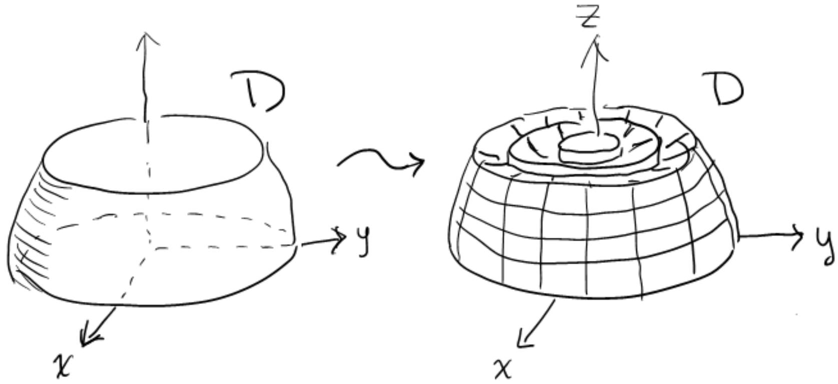
Triple Integrals with Spherical Coordinates

Consider a "spherical box" centered at (ρ, ϕ, θ). Has thickness of $\Delta\rho$ in direction of ρ (away from origin) and other sides are bounded by the angular changes of $\Delta\phi$ (change in ϕ) and $\Delta\theta$ (change in θ).



Volume is $\Delta V =$
$(\Delta\rho)(\rho\Delta\phi)(\rho\sin\phi\,\Delta\theta)$
$= \rho^2\sin\phi\,\Delta\rho\,\Delta\phi\,\Delta\theta$

Now a 3-D solid can be covered with a grid of these boxes $B_1, B_2 \ldots B_n$ each containing a sample point $(\rho_k, \phi_k, \theta_k)$ at its center, and having volume $\rho_k^2\sin\phi_k\,\Delta\rho_k\,\Delta\phi_k\,\Delta\theta_k$



$$\boxed{\text{Then}\quad \iiint_D f(\rho,\phi,\theta)\,dV = \lim_{\|P\|\to 0}\sum_{k=1}^{n} f(\rho_k,\phi_k,\theta_k)\,\rho_k^2\sin\phi_k\,\Delta\rho_k\,\Delta\phi_k\,\Delta\theta_k}$$

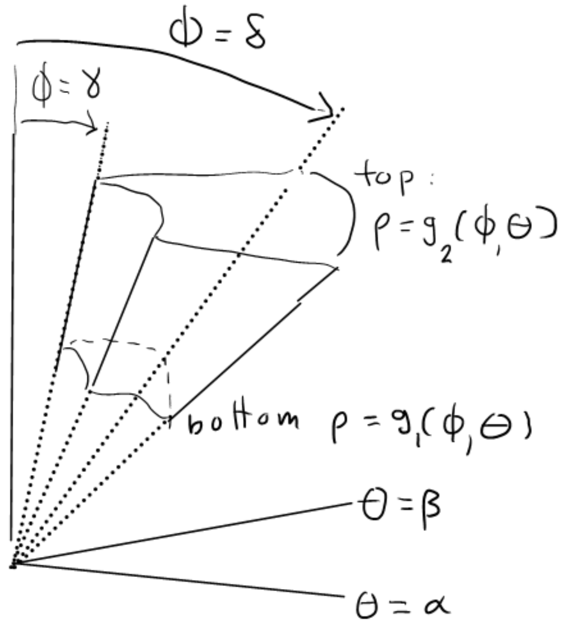
To evaluate such an integral, we use

Fubini's Theorem For Spherical Coordinates

For a region D like the one depicted on the right,

$$\iiint_D f(\rho,\phi,\theta)\,dV =$$

$$\int_\alpha^\beta\int_\gamma^\delta\int_{g_1(\phi,\theta)}^{g_2(\phi,\theta)} f(\rho,\phi,\theta)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$



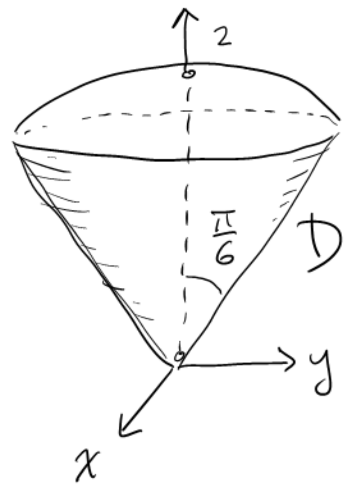
Example

$$D = \{ (\rho, \phi, \theta) \mid 0 \le \rho \le 2,\ 0 \le \phi \le \tfrac{\pi}{6},\ 0 \le \theta \le 2\pi \}$$

This is a conical region cut out from a sphere of radius 2

Suppose its density at (ρ, ϕ, θ) is 4ρ grams per cubic unit.

Find its mass.



$$\text{Mass} = \iiint_D 4\rho \, dV$$

$$= \int_0^{2\pi} \int_0^{\pi/6} \int_0^2 4\rho \ \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/6} \int_0^2 4\rho^3 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/6} \Big[\rho^4 \sin\phi \Big]_0^2 \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\frac{\pi}{6}} 16 \sin\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \Big[-16 \cos\phi \Big]_0^{\frac{\pi}{6}} d\theta = \int_0^{2\pi} -16 \frac{\sqrt{3}}{2} + 16 \, d\theta$$

$$= 16\left(1 - \frac{\sqrt{3}}{2}\right) \int_0^{2\pi} d\theta = 16\left(1 - \frac{\sqrt{3}}{2}\right) 2\pi$$

$$= (32 - 16\sqrt{3})\pi$$

$$= \boxed{16\pi(2 - \sqrt{3}) \text{ grams}}$$