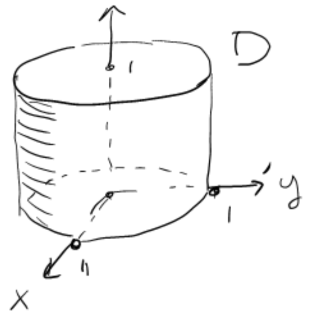
Section 15.7 Triple Integrals in Cylindrical and Spherical Coordinates

Some integrals involve regions and/or functions that have certain symmetries about the z-axis. For these it is sometimes convenient — or necessary to use cylindrical coordinates.

Ex



$$\iiint_D x^2+y^2\,dV = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^1 x^2+y^2\,dz\,dy\,dx$$

$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\left[x^2 z + y^2 z\right]_0^1 dy\,dx$$

D is cylinder over unit circle, height 1.

$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x^2+y^2\,dy\,dx$$

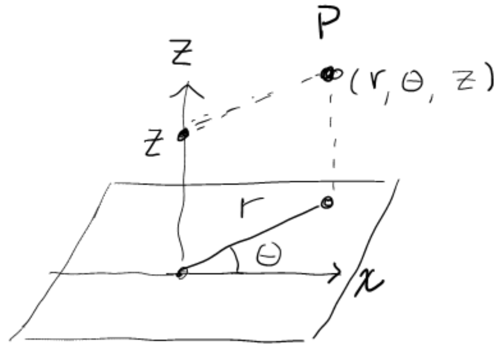
$$= \int_{-1}^{1}\left[x^2 y + \frac{y^3}{3}\right]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx$$

$$= \int_{-1}^{1}\left(2x^2\sqrt{1-x^2} + 2\sqrt{1-x^2}^{\,3}\right)dx$$

$$= \text{(messy and difficult)}$$

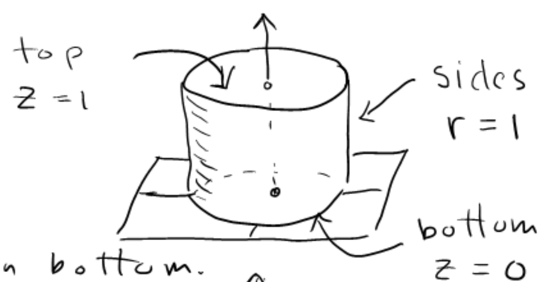
We will revisit this and solve it once we have reviewed cylindrical coordinates.

In cylindrical coordinates a point P in space is described by the triple (r,θ,z), as illustrated

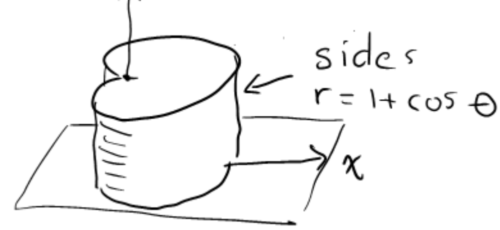


$$\underbrace{(r,\theta,z)}_{\substack{\text{cylindrical}\\ \text{coordinates}\\ \text{of } P}} \longleftrightarrow \underbrace{(r\cos\theta,\ r\sin\theta,\ z)}_{\substack{\text{Cartesian}\\ \text{coordinates}\\ \text{of } P}}$$

Solid D from above example is has simple description in cylindrical coordinates. It's bounded by surface $r=1$ on sides, $z=1$ on top and $z=0$ on bottom.



Of course, regions can be more complex, like this one ⟶

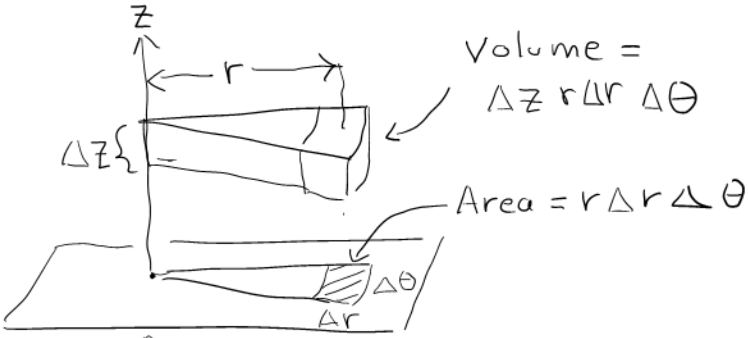


Such a solid D can be the domain of a function $f(r,\theta,z)$. e.g $f(r,\theta,z) = \sqrt{r}\sin(\theta z)$

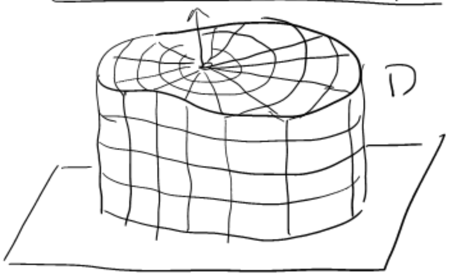
Triple Integrals in Cylindrical Coordinates

We are going to divide D into small "boxes" with polar base $\Delta r \times \Delta\theta$ at distance r from z-axis, and with height Δz.

Box has volume $\Delta z \, r \, \Delta r \, \Delta\theta$



Denote boxes as $B_1 \; B_2 \cdots B_n$ and say box B_k is at distance r_k from z-axis.



B_k has dimensions $\Delta z_k \; \Delta r_k \; r_k \; \Delta\theta_k$ and volume $\Delta V_k = \Delta z_k \, r \, \Delta r_k \, \Delta\theta_k$.

Also each B_k contains a sample point (r_k, θ_k, z_k)

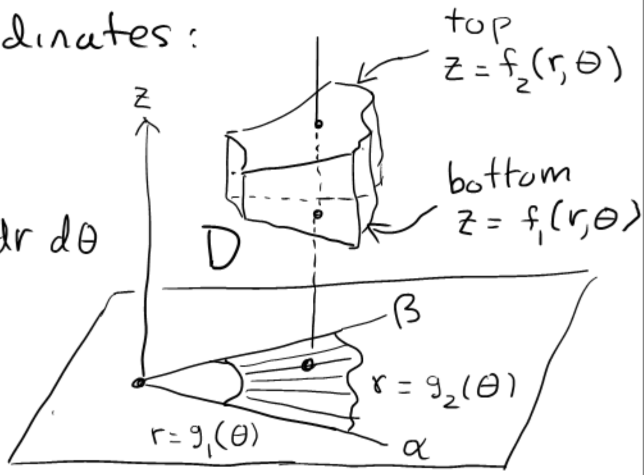
Now suppose function $f(r,\theta,z)$ has domain D.

Define $$\iiint_D f(r,\theta,z)\,dV = \lim_{|P|\to 0} \sum_{k=1}^{n} f(r_k, \theta_k, z_k)\, \Delta z_k \, r \, \Delta r_k \, \Delta\theta_k$$

Fubini's Theorem for cylindrical coordinates:

$$\iiint_D f(r,\theta,z)\,dV = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} \int_{f_1(r,\theta)}^{f_2(r,\theta)} f(r,\theta,z)\,dz\, r\,dr\,d\theta$$

Note the r !!



Note: Often it happens that $g_1(\theta) = 0$.

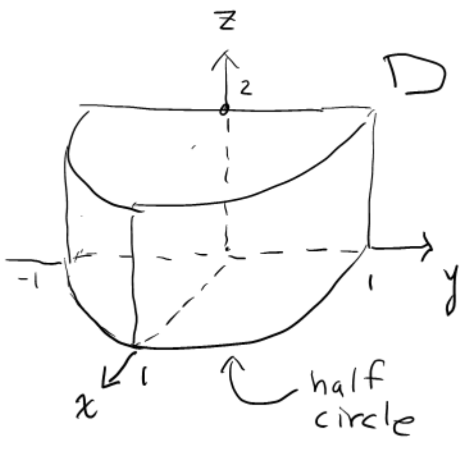
Example

$$\iiint_D r^2\sin^2\theta + z^2\,dV = \int_{-\pi/2}^{\pi/2}\int_0^1\int_0^2 r^2\sin^2\theta + z^2\,dz\,r\,dr\,d\theta$$

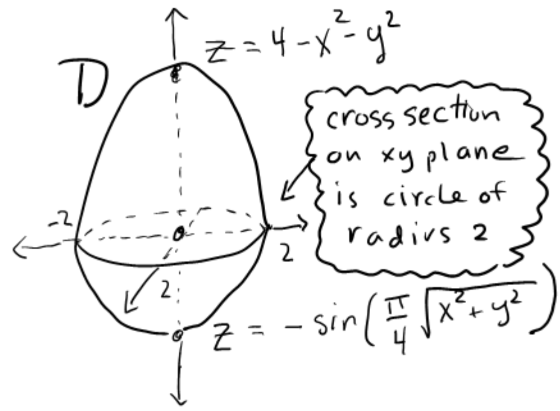
$$= \int_{-\pi/2}^{\pi/2}\int_0^1 \left[r^2\sin^2\theta\, z + \frac{z^3}{3}\right]_0^2 r\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\int_0^1 2r^3\sin^2\theta + \frac{8}{3}r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\left[\frac{r^4\sin\theta}{2} + \frac{4}{3}r^2\right]_0^1 d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \frac{\sin\theta}{2} + \frac{4}{3}\,d\theta = \left[\frac{-\cos\theta}{2} + \frac{4}{3}\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \boxed{\frac{4\pi}{3}}$$



Example Find the volume of this solid.



Because of its z-axis symmetry it looks as if cylindrical coordinates would work well.

Then: $x^2 + y^2 = (r\cos\theta)^2 + (r\sin\theta)^2 = r^2$

Top: $z = 4 - r^2$ Bottom $z = -\cos\left(\frac{\pi}{4} r\right)$

$$V = \iiint_D dV = \int_0^{2\pi}\int_0^2\int_{-\cos(\frac{\pi}{4}r)}^{4-r^2} dz\, r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^2 \left[z\right]_{-\cos(\frac{\pi}{4}r)}^{4-r^2} r\, dr\, d\theta = \int_0^{2\pi}\int_0^2 4r - r^3 + r\cos\left(\frac{\pi}{4}r\right) dr\, d\theta$$

$$= \int_0^{2\pi}\left[2r^2 - \frac{r^4}{4} + \frac{4}{\pi} r\sin\left(\frac{\pi}{4}r\right) + \frac{16}{\pi^2}\cos\left(\frac{\pi}{4}r\right)\right]_0^2 d\theta$$

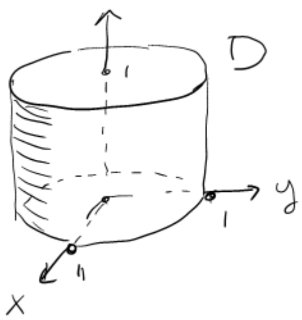
Integration by parts: $\int r\cos\left(\frac{\pi}{4}r\right) dr = \frac{4}{\pi} r\sin\left(\frac{\pi}{4}r\right) + \frac{16}{\pi^2}\cos\left(\frac{\pi}{4}r\right)$

$$= \int_0^{2\pi} 8 - 4 + \frac{4}{\pi}2\sin\frac{\pi}{2} + \frac{16}{\pi^2}\cos\frac{\pi}{2}\, d\theta = \int_0^{2\pi} 4 + \frac{8}{\pi}\, d\theta = \left(4 + \frac{8}{\pi}\right)\int_0^{2\pi} d\theta = \left(4 + \frac{8}{\pi}\right)\left[\theta\right]_0^{2\pi}$$

$$= \boxed{8\pi + 16 \text{ cubic units}}$$

Now let's return to our first problem of the day, the one we couldn't evaluate:

Example



D is cylinder over unit circle, height 1.

$$\iiint_D x^2 + y^2\, dV = ???$$

Idea Change to cylindrical coordinates. D bounded by $r = 1$ on sides and $z = 1$ on top and $z = 0$ on bottom.

$$\iiint_D x^2 + y^2\, dV = \iiint_D (r\cos\theta)^2 + (r\sin\theta)^2\, dV$$

$$= \iiint_D r^2(\cos^2\theta + \sin^2\theta)\, dV = \iiint_D r^2\, dV = \int_0^{2\pi}\int_0^1\int_0^1 r^2\, dz\, r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1\int_0^1 r^3\, dz\, dr\, d\theta = \int_0^{2\pi}\int_0^1 \left[r^3 z\right]_0^1 dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 r^3\, dr\, d\theta = \int_0^{2\pi}\left[\frac{r^4}{4}\right]_0^1 d\theta = \int_0^{2\pi}\frac{1}{4}\, d\theta = \left[\frac{\theta}{4}\right]_0^{2\pi} = \boxed{\frac{\pi}{2}}$$