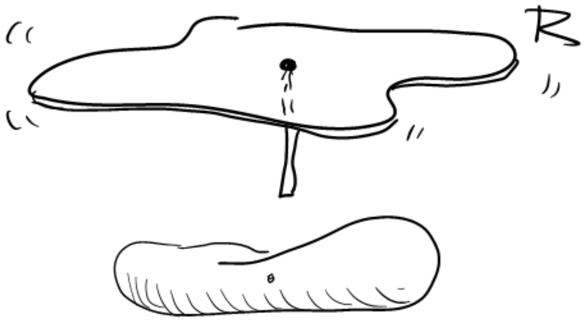
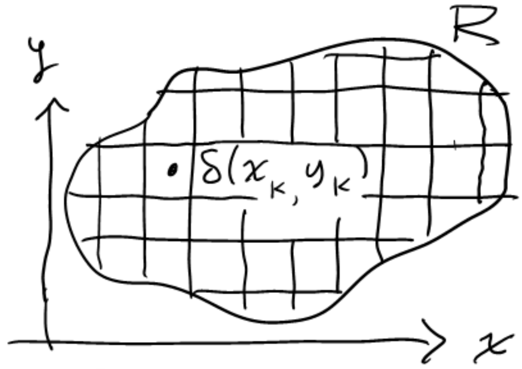
Section 15.6 Moments and Centers of Mass



Goal Determine the center of gravity of a flat plate, that is the point at which it would balance on the tip of a pencil. Also compute the center of gravity of a 3-D solid.

Mass of a Plate



Suppose a flat plate R has density $\delta(x,y)$ grams per square unit at point (x,y).

What is total mass of plate?

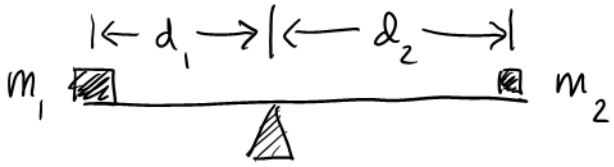
Divide R up into rectangles. $R_1\ R_2\ \ldots\ R_n$.
In each rectangle R_k, put a sample point (x_k, y_k).

$$\begin{pmatrix}\text{mass of}\\ \text{rectangle } k\end{pmatrix} = (\text{density})(\text{area}) \approx \delta(x_k, y_k)\,\Delta x_k\,\Delta y_k$$

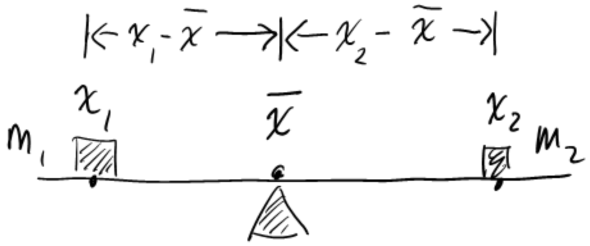
$$\text{Mass} \approx \sum_{k=1}^{n} \delta(x_k\, y_k)\,\Delta x_k\,\Delta y_k$$

$$\boxed{\text{Mass} = \lim_{\|P\|\to 0} \sum_{k=1}^{n} \delta(x_k, y_k)\,\Delta x_k\,\Delta y_k = \iint_R \delta(x,y)\,dA}$$

Now, let's turn our attention to centers of mass.

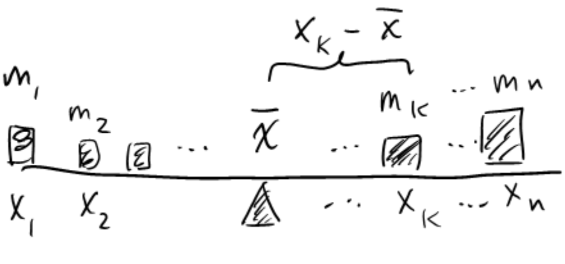


Masses m_1 and m_2 balance if $m_1 d_1 = m_2 d_2$



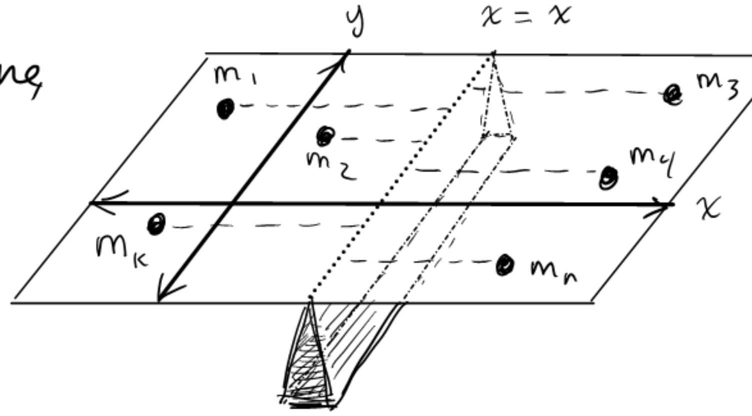
Masses m_1 and m_2 on number line balance at $\bar{x}$ if $(x_1 - \bar{x})m_1 + (x_2 - \bar{x})m_2 = 0$

i.e. $\sum_{k=1}^{2} (x_k - \bar{x})m_k = 0$



Masses $m_1, m_2, \ldots, m_n$ balance at $\bar{x}$ if $\sum_{k=1}^{n} (x_k - \bar{x})m_k = 0$

Now imagine masses $m_1, m_2 \cdots m_n$ are distributed on the xy-plane and any given mass m_k is at the point (x_k, y_k).

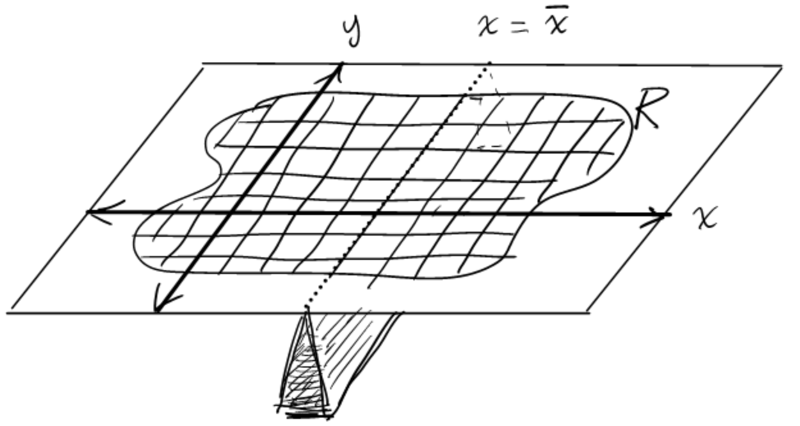


System will balance on the line $x = \bar{x}$ provided that

$$\sum_{k=1}^{n} (x_k - \bar{x}) m_k = 0$$

Next, we are going to compute the vertical line $x = \bar{x}$ that a region R balances on

Consider region R whose density at (x, y) is $\delta(x, y)$
Divide into n rectangles
Put sample point (x_k, y_k) in R_k
Then R_k has mass $\delta(x_k, y_k)\Delta x_k \Delta y_k$.
Region will balance at line $x = \bar{x}$ provided that



$$\sum_{k=1}^{n} (x_k - \bar{x}) \delta(x_k, y_k) \Delta x_k \Delta y_k \approx 0$$

$$\iint_R (x - \bar{x})\, \delta(x, y)\, dA = 0$$

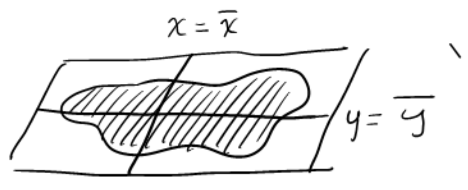
$$\iint_R x\, \delta(x, y)\, dA - \iint \bar{x}\, \delta(x, y)\, dA = 0$$

$$\bar{x} = \frac{\iint_R x\, \delta(x, y)\, dA}{\iint_R \delta(x, y)\, dA} = \frac{M_y}{M}$$

Region R balances on the vertical line $x = \bar{x}$ for this particular $\bar{x}$

Similar Calculations show that region R balances on the <u>horizontal</u> line $y = \bar{y}$, where

$$\bar{y} = \frac{\iint_R y\, \delta(x, y)\, dA}{\iint_R \delta(x, y)\, dA} = \frac{M_x}{M}$$



Region balances at the intersection of these lines, i.e at point $(\bar{x}, \bar{y})$

Summary

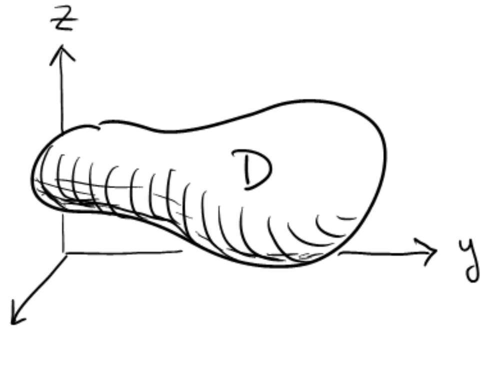
Suppose a 2-D plate has density $\delta(x,y)$ at (x,y).
Then:

$$\text{Mass} = M = \iint_R \delta(x,y)\, dA$$

"First moments" $\begin{cases} M_y = \iint_R x\, \delta(x,y)\, dA \\ M_x = \iint_R y\, \delta(x,y)\, dA \end{cases}$

Center of mass: $(\bar{x}, \bar{y}) = \left(\dfrac{M_y}{M}, \dfrac{M_x}{M}\right)$

Suppose a 3-D solid has density $\delta(x,y,z)$ at (x,y,z)
Then



$$\text{Mass} = M = \iiint_D \delta(x,y,z)\, dV$$

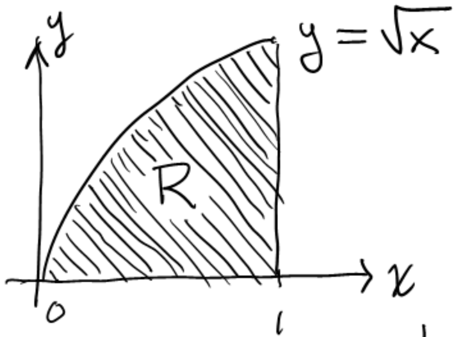
First moments $\begin{cases} M_{yz} = \iiint_D x\, \delta(x,y,z)\, dV \\ M_{xz} = \iiint_D y\, \delta(x,y,z)\, dV \\ M_{yz} = \iiint_D z\, \delta(x,y,z)\, dV \end{cases}$

Center of mass: $(x, y, z) = \left(\dfrac{M_{yz}}{M}, \dfrac{M_{xz}}{M}, \dfrac{M_{xy}}{M}\right)$

Note: Can ignore material on <u>moments</u> <u>of</u> <u>inertia</u>

Example

Find the center of mass of this region, which has uniform density of δ grams per square foot.



$$\text{Mass} = M = \iint_R \delta \, dA = \int_0^1 \int_0^{\sqrt{x}} \delta \, dy \, dx = \int_0^1 \left[\delta y\right]_0^{\sqrt{x}} dx$$

$$= \int_0^1 \delta \sqrt{x} \, dx = \int_0^1 \delta x^{\frac{1}{2}} dx = \left[\delta \frac{2}{3} x^{\frac{3}{2}}\right]_0^1 = \left[\frac{2\delta\sqrt{x}^3}{3}\right]_0^1 = \frac{2\delta}{3} \text{ grams}$$

$$M_y = \iint_R \delta x \, dA = \int_0^1 \int_0^{\sqrt{x}} \delta x \, dy \, dx = \int_0^1 \left[\delta x y\right]_0^{\sqrt{x}} dx$$

$$= \int_0^1 \delta x \sqrt{x} \, dx = \int_0^1 \delta x^{3/2} dx = \left[\delta \frac{2}{5} x^{\frac{5}{2}}\right]_0^1$$

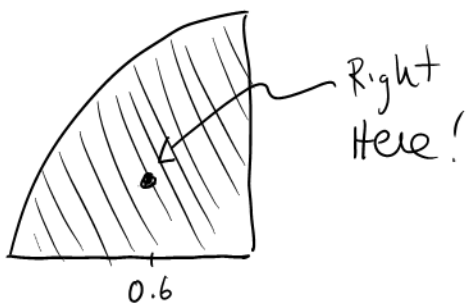
$$= \left[\delta \frac{2}{5} \sqrt{x}^5\right]_0^1 = \boxed{\frac{2\delta}{5}}$$

$$M_x = \iint_R \delta y \, dA = \int_0^1 \int_0^{\sqrt{x}} \delta y \, dy \, dx = \int_0^1 \left[\delta \frac{y^2}{2}\right]_0^{\sqrt{x}} dx$$

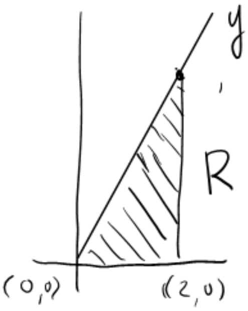
$$= \int_0^1 \delta \frac{x}{2} dx = \left[\delta \frac{x^2}{4}\right]_0^1 = \boxed{\frac{\delta}{4}}$$

Center of mass: $(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(\frac{\frac{2\delta}{5}}{\frac{2\delta}{3}}, \frac{\frac{\delta}{4}}{\frac{2\delta}{3}}\right)$

$$= \left(\frac{3}{5}, \frac{3}{8}\right) = \boxed{(0.6, \; 0.375)}$$



Example [Time permitting]



Density $= \delta(x,y) = 4x + 2y + 2$ grams / sq unit at point (x,y). Find mass and center of mass $(\bar{x}, \bar{y})$.

$$M = \iint_R (4x+2y+2)\,dA = \int_0^2 \int_0^{2x} 4x+2y+2 \, dy\, dx$$

$$= \int_0^2 \Big[4xy + y^2 + 2y\Big]_0^{2x} dx = \int_0^2 12x^2 + 4x \, dx$$

$$= \Big[4x^3 + 2x^2\Big]_0^2 = 32 + 8 = \boxed{40 \text{ grams}}$$

$$M_y = \iint_R x(4x+2y+2)\,dA = \int_0^2 \int_0^{2x} 4x^2 + 2xy + 2x \, dy\, dx$$

$$= \int_0^2 \Big[4x^2 y + xy^2 + 2xy\Big]_0^{2x} dx = \int_0^2 12x^3 + 4x^2 \, dx$$

$$= \Big[3x^4 + \frac{4}{3}x^3\Big]_0^2 = 3 \cdot 16 + \frac{4}{3} \cdot 8 = 48 + \frac{32}{3} = \boxed{\frac{176}{3}}$$

$$M_y = \iint_R y(4x+2y+2)\,dA = \int_0^2 \int_0^{2x} 4xy + 2y^2 + 2y \, dy\, dx$$

$$= \int_0^2 \Big[2xy^2 + \frac{2}{3}y^3 + y^2\Big]_0^{2x} dx = \int_0^2 8x^3 + \frac{16}{3}x^3 + 4x^2 \, dx$$

$$= \int_0^2 \frac{40}{3}x^3 + 4x^2 \, dx = \Big[\frac{10}{3}x^4 + \frac{4}{3}x^3\Big]_0^2$$

$$= \frac{160}{3} + \frac{32}{3} = \boxed{\frac{192}{3}}$$

Center of mass: $\left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(\frac{\frac{176}{3}}{40}, \frac{\frac{192}{3}}{40}\right)$

$$= \left(\frac{22}{15}, \frac{8}{5}\right)$$