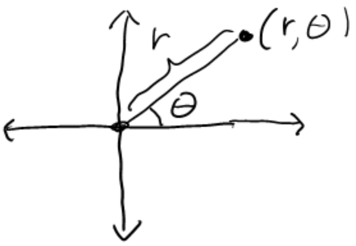
Section 15.4 Double Integrals in Polar Form

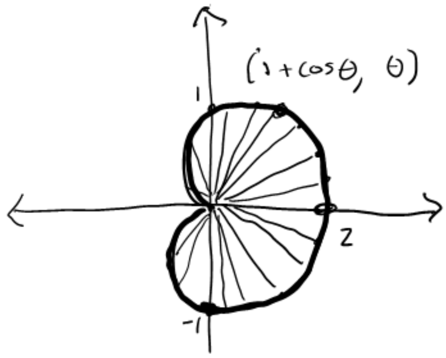
Recall the basics of polar coordinates



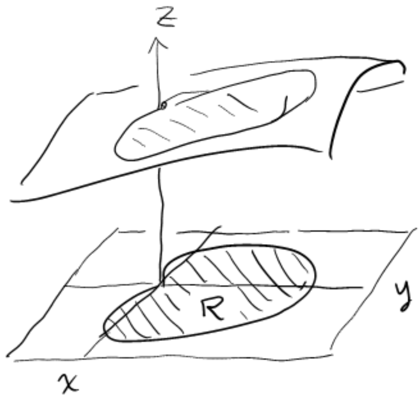
Any point on plane is located via a polar coordinate (r, θ).

$$(r, \theta) \longleftrightarrow (r\cos\theta, r\sin\theta)$$

polar coordinate of a point — Cartesian coordinate of that same point



Graph of $r = 1 + \cos\theta$



$$z = f(x, y) = f(r\cos\theta, r\sin\theta) = F(r, \theta)$$

Often this is simpler than $f(x,y)$, especially when R and the graph of $z = f(x,y)$ have some kind of polar symmetry (i.e. symmetry about z-axis "pole")

Todays Goals

1. Define $\iint_R f(r, \theta)\, dA$
2. Learn how to compute this integral
3. Compute $\iint_R f(x,y)\, dA$ by converting to $\iint_R f(r\cos\theta, r\sin\theta)\, dA$

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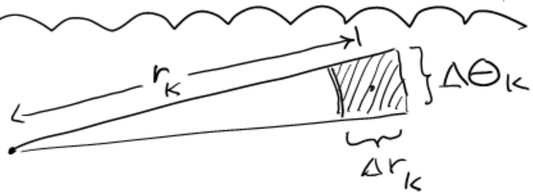
① Defining $\iint_R f(r, \theta)\, dA$:

Area of a "polar rectangle"

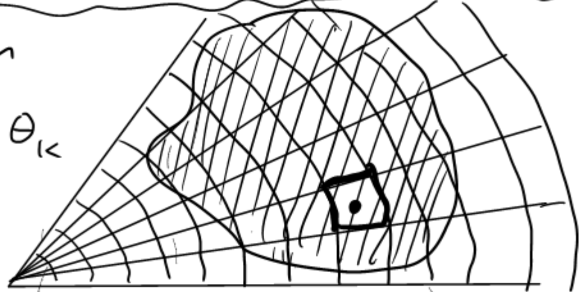


Area $= \Delta A_k = r_k \Delta r_k \Delta\theta_k$

Cover R with a grid of polar rectangles.
Rectangles inside R are $R_1, R_2, R_3, \ldots R_n$

Rectangle R_k has area $\Delta A_k = r_k \Delta r_k \Delta \theta_k$

Put sample point (r_k, θ_k) in each R_k



With all this, we make the following definition:

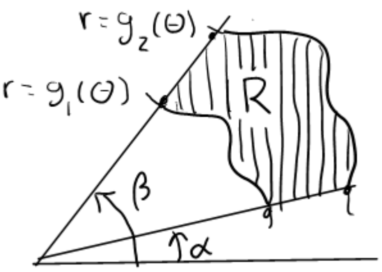
$$\iint_R f(r, \theta)\, dA = \lim_{\|P\| \to 0} \sum_{k=1}^{n} f(r_k, \theta_k)\, r_k \Delta r_k \Delta\theta_k$$

The form we just derived gives some indication of how to compute the integral

$$\iint_R f(r,\theta)\,dA = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(r_i, \theta_i)\, r_k\, \Delta r_k\, \Delta\theta_k$$

$$= \int_?^? \int_?^? f(r\,\theta)\, r\, dr\, d\theta$$

Fubini's Theorem in Polar Form

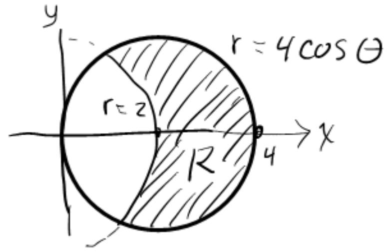


$$\iint_R f(r,\theta)\,dA = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r,\theta)\, r\, dr\, d\theta$$

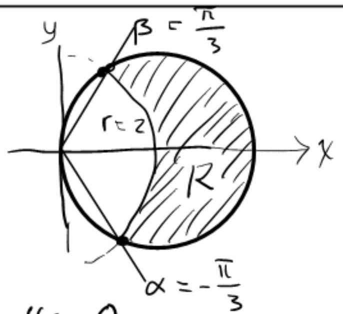
Note: Area of R $= \iint_R 1\,dA = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} r\,dr\,d\theta$

Example

Find $\iint_R \frac{\theta}{r}\,dA$



Solution First find what θ values enclose R. Solve $2 = 4\cos\theta$. Get $\theta = \pm\frac{\pi}{3}$



$$\iint_R \frac{\theta}{r}\,dA = \int_{-\pi/3}^{\pi/3} \int_2^{4\cos\theta} \frac{\theta}{r}\, r\,dr\,d\theta = \int_{-\pi/3}^{\pi/3} \int_2^{4\cos\theta} \theta\,dr\,d\theta = \int_{-\pi/3}^{\pi/3} \Big[\theta r\Big]_2^{4\cos\theta} d\theta$$

$$= \int_{-\pi/3}^{\pi/3} 4\theta\cos\theta - 2\theta\; d\theta = \Big[4\theta\sin\theta + 4\cos\theta - \theta^2\Big]_{-\pi/3}^{\pi/3}$$ ← Integration by parts

$$= \left(4\frac{\pi}{3}\sin\frac{\pi}{3} + 4\cos\frac{\pi}{3} - \left(\frac{\pi}{3}\right)^2\right) - \left(4\left(-\frac{\pi}{3}\right)\sin -\frac{\pi}{3} + 4\cos -\frac{\pi}{3} - \left(-\frac{\pi}{3}\right)^2\right) = \boxed{\frac{4\pi\sqrt{3}}{3}}$$

Example Find the area of the above region.

$$A = \int_{-\pi/3}^{\pi/3} \int_2^{4\cos\theta} r\,dr\,d\theta = \int_{-\pi/3}^{\pi/3} \left[\frac{r^2}{2}\right]_2^{4\cos\theta} d\theta = \int_{-\pi/3}^{\pi/3} (8\cos^2\theta - 2)\,d\theta$$

$$= \int_{-\pi/3}^{\pi/3} 8\,\frac{1+\cos 2\theta}{2} - 2\;d\theta = \int_{-\pi/3}^{\pi/3} 2 + 4\cos 2\theta\;d\theta = \Big[2\theta + 2\sin 2\theta\Big]_{-\pi/3}^{\pi/3}$$

$$= \left(\frac{2\pi}{3} + 2\sin\frac{2\pi}{3}\right) - \left(-\frac{2\pi}{3} + 2\sin -\frac{2\pi}{3}\right) = \frac{2\pi}{3} + 2\frac{\sqrt{3}}{2} + 2\frac{\sqrt{3}}{2} + \frac{2\pi}{3}$$

$$= \boxed{2\sqrt{3} + \frac{4\pi}{3} \text{ square units}}$$

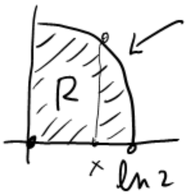
③ Changing Cartesian Integrals into Polar integrals.

Recall $(r,\theta) \longleftrightarrow (r\cos\theta,\ r\sin\theta)$

$\uparrow$ polar $\qquad$ $\uparrow$ Cartesian

Example

Evaluate $\displaystyle\iint_R e^{\sqrt{x^2+y^2}}\, dA$



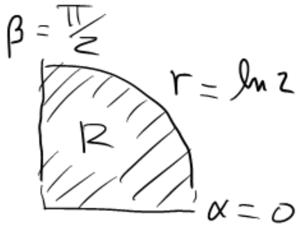
quarter circle of radius $\ln 2$

$y = \sqrt{(\ln 2)^2 - x^2}$

Solution $\displaystyle\int_0^{\ln 2}\int_0^{\sqrt{(\ln 2)^2 - x^2}} e^{\sqrt{x^2+y^2}}\, dy\, dx$ ⟸ unpleasant integral

Idea: Switch to polar form

Then $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$



$$\iint_R e^{\sqrt{x^2+y^2}}\, dA = \iint_R e^{\sqrt{(r\cos\theta)^2 + (r\sin\theta)^2}}\, dA$$

$$= \iint_R e^{\sqrt{r^2(\cos^2\theta + \sin^2\theta)}}\, dA$$

$$= \iint_R e^r\, dA = \int_0^{\frac{\pi}{2}}\int_0^{\ln 2} e^r\, r\, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \Big[re^r - e^r\Big]_0^{\ln 2}\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(\ln 2\, e^{\ln 2} - e^{\ln 2}\right) - \left(0e^0 - e^0\right)\, d\theta$$

$$= \int_0^{\pi/2} (2\ln 2 - 2 + 1)\, d\theta$$

$$= (2\ln 2 - 1)\int_0^{\frac{\pi}{2}} d\theta = \boxed{\frac{\pi}{2}(2\ln 2 - 1)}$$

Integration by Parts

$\int e^r r\, dr$

$= \int r e^r\, dr$

$u = r \longrightarrow du = dr$

$dv = e^r dr \rightarrow v = e^r$

$\int re^r dr = uv - \int v\, du$

$= re^r - \int e^r dr$

$= re^r - e^r$