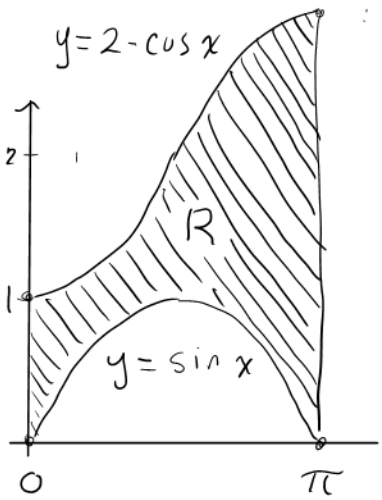
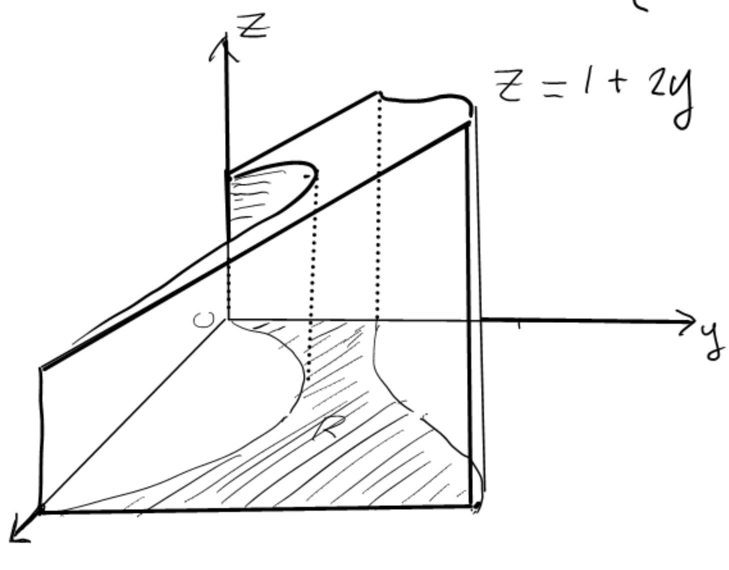
Section 15.2 Double Integrals over Regions (continued)

We will round out our discussion with one more example. It requires the following trig identities.

$$\begin{cases} \sin^2\theta = \frac{1}{2}(1-\cos 2\theta) \\ \cos^2\theta = \frac{1}{2}(1+\cos 2\theta) \end{cases}$$





Find the volume of the solid over R and below the graph of $z = f(x,y) = 1 + 2y$.

$$V = \iint_R 1+2y \, dA = \int_0^{\pi}\int_{\sin x}^{2-\cos x} (1+2y)\,dy\,dx = \int_0^{\pi} \Big[y + y^2 \Big]_{\sin x}^{2-\cos x} dx$$

$$= \int_0^{\pi} \left((2-\cos x) + (2-\cos x)^2 - \sin x - (\sin x)^2 \right) dx$$

$$= \int_0^{\pi} \left(2 - \cos x + 4 - 4\cos x + \cos^2 x - \sin x - \sin^2 x \right) dx$$

$$= \int_0^{\pi} \left(6 - 5\cos x + \frac{1+\cos 2x}{2} - \sin x - \frac{1-\cos 2x}{2} \right) dx$$

$$= \int_0^{\pi} \left(6 - 5\cos x - \sin x + \cos 2x \right) dx$$

$$= \Big[6x - 5\sin x + \cos x + \tfrac{1}{2}\sin 2x \Big]_0^{\pi}$$

$$= \left(6\pi - 5\sin\pi + \cos\pi + \tfrac{1}{2}\sin\pi\right) - \left(6\cdot 0 - 5\sin 0 + \cos 0 + \tfrac{1}{2}\sin 0\right)$$

$$= (6\pi - 0 - 1 + 0) - (0 - 0 + 1 + 0) = \boxed{6\pi - 2 \text{ cubic units}}$$

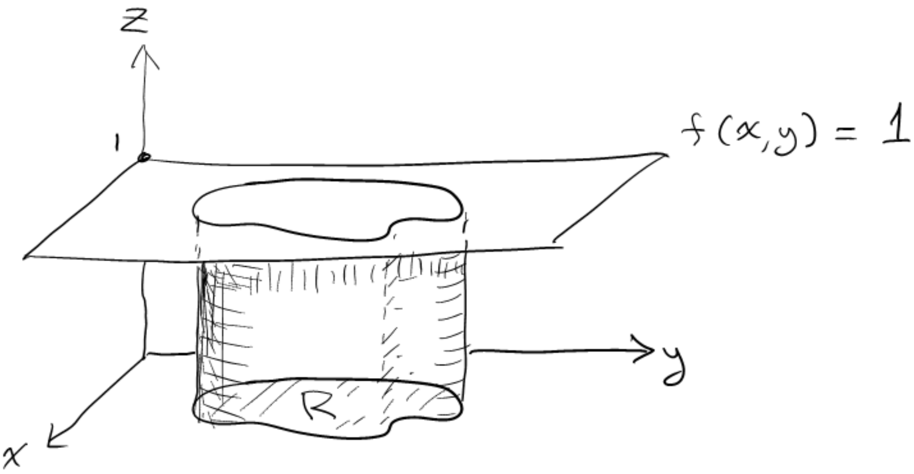
Section 15.3 Area by Double Integration

Consider the volume over a region R and under the graph of $z = f(x,y) = 1$.

This is a cylinder of base R and height 1.

Its volume is therefore

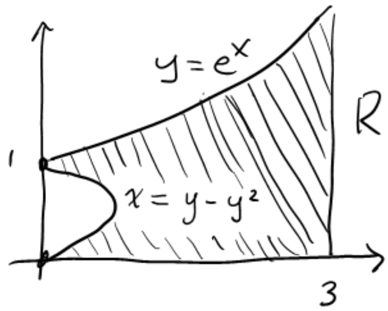
$$V = (\text{area of } R) \cdot 1$$
$$= (\text{area of } R)$$

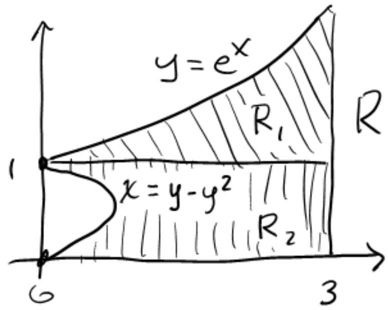


But also $V = \iint_R f(x,y)\,dA = \iint_R 1\,dA = \iint_R dA$

Therefore $\boxed{(\text{Area of a region } R) = \iint_R dA}$

Example Find the volume





Divide into two regions as shown.

$$\text{Area} = \iint_R dA = \iint_{R_1} dA + \iint_{R_2} dA$$

$$= \int_0^3 \int_1^{e^x} dy\,dx + \int_0^1 \int_{y-y^2}^{3} dx\,dy$$

$$= \int_0^3 \Big[y\Big]_1^{e^x} dx + \int_0^1 \Big[x\Big]_{y-y^2}^{3} dy$$

$$= \int_0^3 (e^x - 1)\,dx + \int_0^1 (3 - y + y^2)\,dy$$

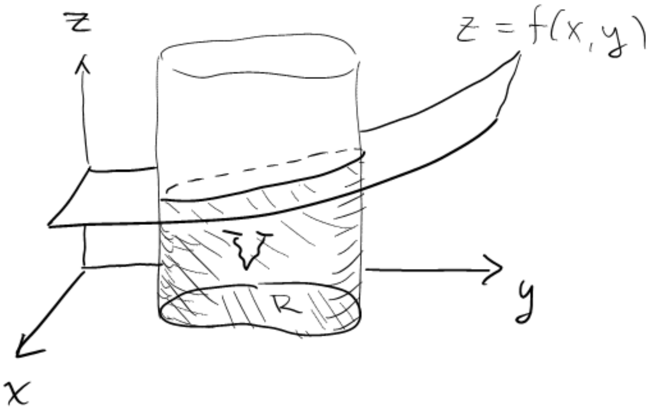
$$= \Big[e^x - x\Big]_0^3 + \Big[3y - \frac{y^2}{2} + \frac{y^3}{3}\Big]_0^1$$

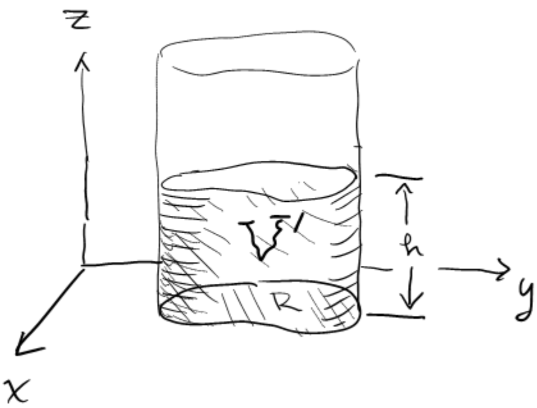
$$= e^3 - 3 - (e^0 - 0) + 3 - \frac{1}{2} + \frac{1}{3} = \boxed{e^3 - \frac{7}{6} \text{ sq. units}}$$

$$= e^3 - 3 - 1 + 0 + 3 - \tfrac{1}{2} + \tfrac{1}{3}$$

Average Value

Question: What is the average value of $f(x, y)$ on the region R?





To answer this think of the solid region under $z = f(x,y)$ and above R as a liquid in a "glass" over R, meeting the graph.

No let the liquid settle & level out to uniform dept of h.

$$\left(\text{Average value of } z = f(x,y)\right) = \left(\text{average depth of liquid}\right) = h$$

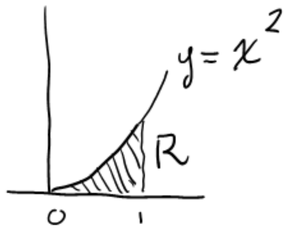
Since the volumes of the two solids above are equal, we get

$$V = V'$$

$$\iint_R f(x,y)\,dA = \iint_R h\,dA = h \iint_R dA$$

$$\Longrightarrow \quad \boxed{\begin{pmatrix}\text{Average Value} \\ \text{of } f(x,y) \text{ over} \\ \text{region } R\end{pmatrix} = h = \frac{\iint_R f(x,y)\,dx}{\iint_R dA} = \frac{\iint_R f(x,y)\,dA}{\text{area of } R}}$$

Example Find the average value of $f(x,y) = x^2y + x$ over this region:



$$\text{Ave} = \frac{\iint_R x^2y + x\,dA}{\iint_R dA} = \frac{\int_0^1 \int_0^{x^2} x^2y + x\,dy\,dx}{\int_0^1 \int_0^{x^2} dy\,dx} = \frac{\int_0^1 \left[\frac{x^2y}{2} + xy\right]_0^{x^2} dx}{\int_0^1 \left[y\right]_0^{x^2} dx}$$

$$= \frac{\int_0^1 \frac{x^4}{2} + x^3\,dx}{\int_0^1 x^2\,dx} = \frac{\left[\frac{x^5}{10} + \frac{x^4}{4}\right]_0^1}{\left[\frac{x^3}{3}\right]_0^1} = \frac{\frac{1}{10} + \frac{1}{4}}{\frac{1}{3}} = \frac{3}{10} + \frac{3}{4}$$

$$= \frac{12}{40} + \frac{30}{40} = \frac{42}{40} = \boxed{\frac{21}{20}}$$