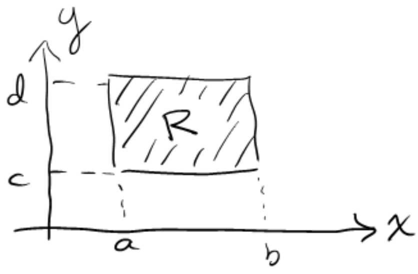
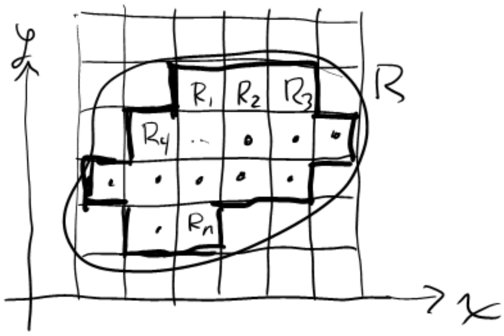
Section 15.2 Double Integrals over General Regions

Recall



- $\displaystyle\iint_R f(x,y)\,dA = \lim_{|P|\to 0} \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k$
- $\displaystyle\iint_R f(x,y)\,dA = \int_a^b\int_c^d f(x,y)\,dy\,dx = \int_c^d\int_a^b f(x,y)\,dx\,dy$
- $\displaystyle\iint_R f(x,y)\,dA$ = volume under $z = f(x,y)$, above R (if $f(x,y) \geq 0$)

Today's Goal Make this work for non-rectangular (general) regions.



Definitions Given a region R on the plane, cover it with a grid. Label the grid rectangles that lie entirely inside the region as R_1 R_2 R_3 ... R_n.

- Say R_k has dimensions $\Delta x_k \times \Delta y_k$
- Let $|P|$ be the diagonal of the largest rectangle. Thus if $|P| \to 0$, then $n \to \infty$.
- Put a sample point (x_k, y_k) inside each R_k
- Let $\Delta A_k = \Delta x_k\,\Delta y_k$ = (area of k^{th} rectangle)
- Riemann sum: $\displaystyle\sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k$

Definition The definite integral of $f(x,y)$ over R is

$$\iint_R f(x,y)\,dA = \lim_{|P|\to 0} \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k.$$

(Provided the limit exists – if it does we say $f(x,y)$ is integrable over R.)

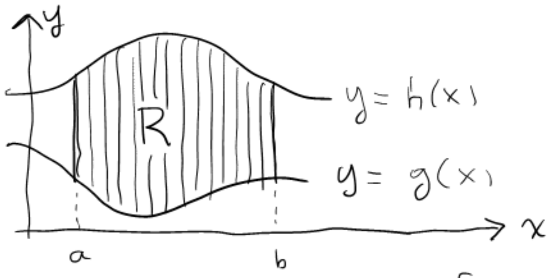
Theorem If $f(x,y)$ is continuous on R then $\displaystyle\iint_R f(x,y)\,dx$ exists.

Next, let's see how we would compute such an integral.

Theorem 2 (Fubini's Theorem)

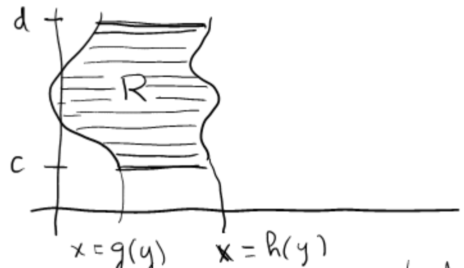
Suppose $f(x,y)$ is continuous on a region R

If R has this form



then $\displaystyle\iint_R f(x,y)\,dA = \int_a^b \int_{g(x)}^{f(x)} f(x,y)\,dy\,dx$

If R has this form

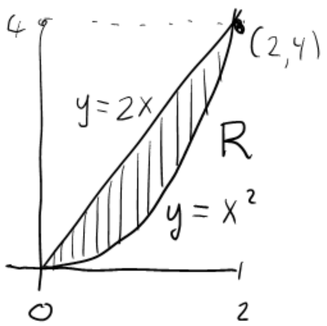


Then $\displaystyle\iint_R f(x,y)\,dA = \int_c^d \int_{g(y)}^{h(y)} f(x,y)\,dx\,dy$

Note: Some regions have neither form: 

More on that later. First, some examples

Example



$f(x,y) = x^3 + 4y$

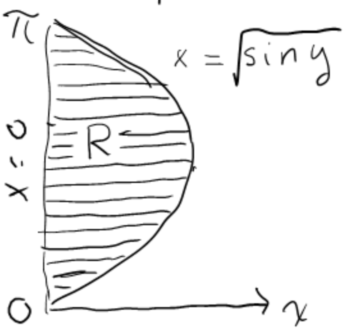
$$\iint_R f(x,y)\,dA = \int_0^2 \int_{x^2}^{2x} (x^3+4y)\,dy\,dx$$

$$= \int_0^2 \Big[x^3 y + 2y^2 \Big]_{x^2}^{2x} dx$$

$$= \int_0^2 \left(x^3\, 2x + 2(2x)^2 \right) - \left(x^3 x^2 + 2(x^2)^2 \right) dx = \int_0^2 2x^4 + 8x^2 - x^5 - 2x^4 \, dx$$

$$= \int_0^2 8x^2 - x^5 \, dx = \left[\frac{8}{3}x^3 - \frac{1}{6}x^6 \right]_0^2 = \frac{8}{3}2^3 - \frac{1}{6}2^6 = \frac{64}{3} - \frac{2^5}{3} = \boxed{\frac{33}{3}}$$

Example



$f(x,y) = xy$

$$\iint_R f(x,y)\,dA = \int_0^{\pi} \int_0^{\sqrt{\sin y}} xy\,dx\,dy$$

$$= \int_0^{\pi} \left[\frac{x^2 y}{2} \right]_0^{\sqrt{\sin y}} dy = \int_0^{\pi} \frac{\sin(y)\,y}{2}\,dy$$

$\int y \sin y \, dy$

$u = y \qquad du = dy$

$dv = \sin y\,dy \qquad v = \int \sin y\,dy = -\cos y$

$\int u\,dv = uv - \int v\,du$

$= -y\cos y + \int \cos y\,dy = -y\cos y + \sin y$

$$= \frac{1}{2}\Big[-y\cos y + \sin y \Big]_0^{\pi}$$

$$\frac{1}{2}\left\{ \left(-\pi\cos\pi + \sin\pi \right) - \left(-0\cos 0 + \sin 0 \right) \right\}$$

$$= \boxed{\frac{\pi}{2}}$$

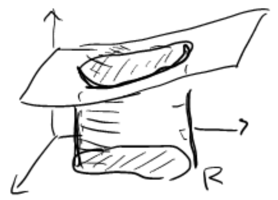
Properties

① $\iint_R c\, f(x,y)\, dA = c \iint_R f(x,y)\, dA$

② $\iint_R \big(f(x,y) \pm g(x,y)\big)\, dA = \iint_R f(x,y)\, dA \pm \iint_R g(x,y)\, dA$

③ If $f(x,y) \geq 0$ on R, then

$$\iint_R f(x,y)\, dA = \begin{pmatrix} \text{volume under } z = f(x,y) \\ \text{and above } R \end{pmatrix} \geq 0$$

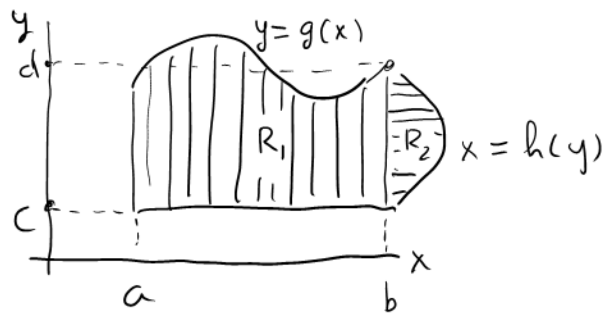


④ If $f(x,y) \geq g(x,y)$ on R, then $\iint_R f(x,y)\, dA \geq \iint_R g(x,y)$

⑤ If 

then $\iint_R f(x,y)\, dA = \iint_{R_1} f(x,y)\, dA + \iint_{R_2} f(x,y)\, dA$

The last property can be useful for regions not of the type for which Fubini's theorem applies:



$$\iint_R f(x,y)\, dx = \iint_{R_1} f(x,y)\, dA + \iint_{R_2} f(x,y)\, dA$$

$$= \int_a^b \int_c^{g(x)} f(x,y)\, dy\, dx + \int_c^d \int_b^{h(y)} f(x,y)\, dx\, dy \quad \text{etc.}$$

Example $\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (2 - x^2 - y^2)\, dy\, dx$ gives the volume of a solid.

Describe the solid.