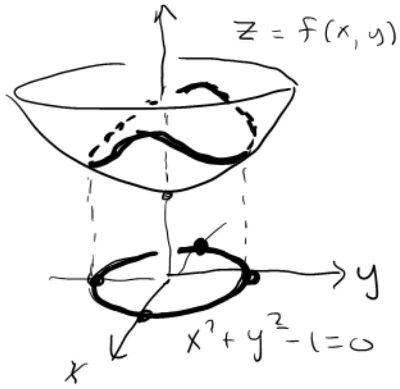
Section 14.8 Lagrange Multipliers (Lagrange 1736 – 1813)

We will now consider what are called "constrained Max / Min problems. This will also give a method for dealing with the boundary of the region in finding absolute max / min on a closed region.

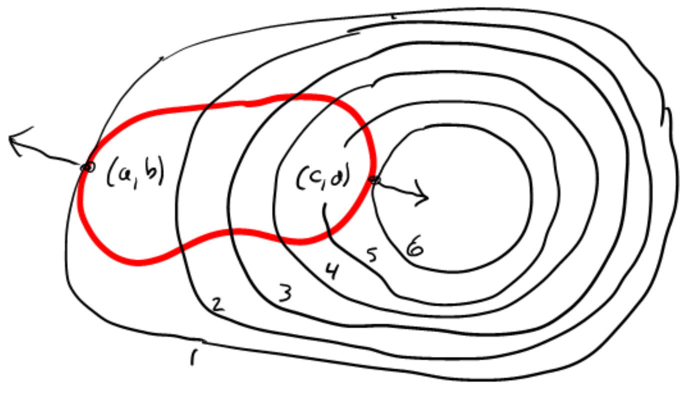


Here is a typical problem

Find the absolute max and min of $z = f(x,y) = x^2 + \frac{y^2}{4} + 2$ subject to the constraint $\underbrace{x^2+y^2-1=0}_{g(x,y)=0}$

From the picture it appears that there's a maximum at $(1,0)$ and $(-1,0)$ and a minimum at $(0,1)$ and $(0,-1)$. But how could we determine such max / min in general?

To see how, consider an arbitrary function $z = f(x,y)$ subject to a constraint $g(x,y)=0$. The level curves might look something as follows.



The constraint $g(x,y)=0$ is also graphed (in red).

There is a min at (a,b), and $\nabla f(a,b) = \lambda \nabla g(a,b)$ for some scalar λ.

There is a max at (c,d), and $\nabla f(c,d) = \lambda \nabla g(c,d)$.

The extrema of $f(x,y)$ subject to the constraint $g(x,y)=0$ happen at (x,y) for which $\nabla f(x,y) = \lambda g(x,y)$ for some constant λ.

Method of Lagrange Multipliers

To find the max and min of $f(x,y)$ subject to the constraint $g(x,y)=0$, solve the system

$$\begin{cases} \nabla f(x,y) = \lambda \nabla g(x,y) \\ g(x,y) = 0 \end{cases}$$

for x, y and λ. Each solution $(x,y)=(a,b)$ will be a potential location of a max or min.

Same method works for functions of 3 or more variables. Number λ (which you throw away at the end) is called a Lagrange multiplier

Example: Find the maximum and minimum values (and their locations) of $z = f(x) = x^2 + \frac{y^2}{4}$ subject to $g(x,y) = x^2 + y^2 - 1 = 0$

$$\begin{cases} \nabla f(x,y) = \lambda \nabla g(x,y) \\ g(x,y) = 0 \end{cases} \Rightarrow \begin{cases} \langle 2x, \frac{y}{2} \rangle = \lambda \langle 2x, 2y \rangle \\ x^2 + y^2 - 1 = 0 \end{cases}$$

We get the system:

$$\begin{cases} 2x = \lambda 2x \\ \frac{y}{2} = \lambda 2y \\ x^2 + y^2 = 1 \end{cases} \Rightarrow \begin{cases} x = \lambda x & (1) \\ y = \lambda 4y & (2) \\ x^2 + y^2 = 1 & (3) \end{cases}$$

If $x \neq 0$, then $\lambda = 1$ (by (1)) and $y = 0$ (by (2)), and $x^2 + 0^2 = 1$ (by (3)), so $x = \pm 1$. Get points $(1,0)$ and $(-1,0)$.

If $x = 0$, then $y = \pm 1$ (by (3)) and $\lambda = \frac{1}{4}$ (by (2))
Get points $(0,1)$ and $(0,-1)$

$f(1,0) = 1$, $f(-1,0) = 1$ } maximum

$f(0,1) = 1/4$, $f(0,-1) = 1/4$ } minimum

Conclusion

$f(x,y)$ subject to constraint $g(x,y) = 0$ has an abs. max at $(1,0)$ and $(-1,0)$ and an abs min at $(0,1)$ and $(0,-1)$

Example

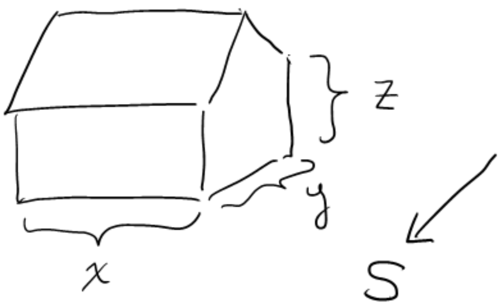
House is to contain 14000 cubic feet (not including attic)

Annual heating costs:

$4 per square foot for floor

$2 per square foot for South wall

$3 per square foot for all other walls.



Problem: Find dimensions x, y, z that minimize heating costs

Heating cost: $4xy + 2xz + 3\cdot 2(yz) + 3xz$

$\rightsquigarrow C(x,y,z) = 4xy + 5xz + 6yz$

Thus we need to minimize

$C(x,y,z) = 4xy + 5xz + 6yz$

Subject to

$$g(x,y,z) = \underbrace{xyz - 14000}_{\text{(volume 14000 cubic ft)}} = 0$$

Note $C(x,y,z)$ has no maximum when $g(x,y,z)=0$.

Reason: Let $z=1$. Then:

$g(x,y,z)=0 \Rightarrow xy - 14000 = 0$

$\Rightarrow y = \frac{14000}{x}$

$C(x,y,z) = 56000 + 5x + \frac{6\cdot 14000}{x}$, and this gets arbitrarily large as $x \to 0$. Thus there will be no maximum. We seek minimum.

Solve:

$$\begin{cases} \nabla C(x,y,z) = \lambda \nabla g(x,y,z) \\ g(x,y,z) = 0 \end{cases}$$

$$\begin{cases} \langle 4y+5z,\ 4x+6z,\ 5x+6y \rangle = \lambda \langle yz,\ xz,\ xy \rangle \\ xyz - 14000 = 0 \end{cases}$$

$$\begin{cases} 4y + 5z = \lambda yz \\ 4x + 6z = \lambda xz \\ 5x + 6y = \lambda xy \\ xyz - 14000 = 0 \end{cases} \Rightarrow \begin{cases} 4xy + 5xz = \lambda xyz & (1) \\ 4xy + 6yz = \lambda xyz & (2) \\ 5xz + 6yz = \lambda xyz & (3) \\ xyz = 14000 & (4) \end{cases}$$

(1) − (2): $5xz - 6yz = 0 \Rightarrow z(5x - 6y) = 0 \Rightarrow \boxed{x = \frac{6}{5}y}$ (5)

(2) − (3): $4xy - 5xz = 0 \Rightarrow x(4y - 5z) = 0 \Rightarrow \boxed{z = \frac{4}{5}y}$ (6)

(4), (5), (6) $\Rightarrow \left(\frac{6}{5}y\right)y\left(\frac{4}{5}y\right) = 14000 \Rightarrow \frac{24}{25}y^3 = 14000$

$\Rightarrow y^3 = \frac{25\cdot 14000}{24} = \frac{5\cdot5\cdot5\cdot8\cdot350}{8\cdot3} = \frac{5^3\cdot 350}{3} \Rightarrow \boxed{y = 5\sqrt[3]{\frac{350}{3}}}$

Thus $x = 6\sqrt[3]{\frac{350}{3}}$, $y = 5\sqrt[3]{\frac{350}{3}}$, $z = 4\sqrt[3]{\frac{350}{3}}$

minimizes total heating cost.