Section 14.5 (Continued)

Recall that the directional derivative of $f(x,y)$ at (x,y) in the direction of the unit vector $\vec{u}$ is

$$D_{\vec{u}} f(x,y) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \cdot \vec{u} = \begin{pmatrix} \text{Rate of change of} \\ z = f(x,y) \text{ at } (x,y) \\ \text{in the direction of } \vec{u} \end{pmatrix}$$

The expression $\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$ that occurs in this expression is significant. It is called the <u>gradient vector</u> of f.

> <u>Definition</u> The <u>gradient</u> of a function $f(x,y)$ is the (variable) vector $\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$ on the xy-plane.

<u>Example</u>: $f(x,y) = xy$ $\qquad \nabla f = \langle y, x \rangle$

Notice that ∇f is a vector that depends on x any y so it makes sense to write it as

$$\nabla f(x,y) = \langle y, x \rangle.$$

Thus at any point (x,y) on the xy-plane there is a corresponding gradient vector $\nabla f(x,y) = \langle y, x \rangle$.

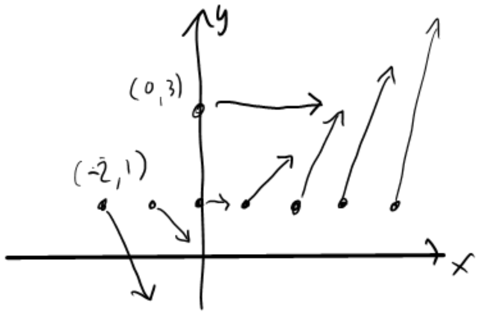
$\nabla f(1,1) = \langle 1, 1 \rangle$

$\nabla f(1,2) = \langle 2, 1 \rangle$

$\nabla f(1,3) = \langle 3, 1 \rangle$

$\nabla f(0,3) = \langle 3, 0 \rangle$

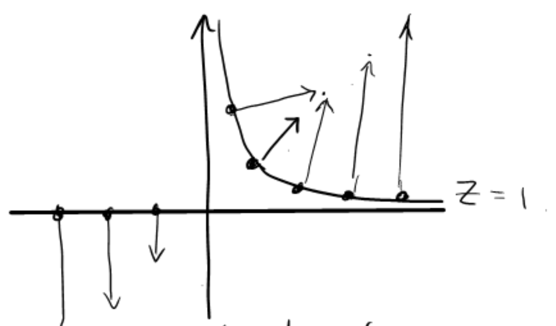
etc., as pictured.



To understand the rhyme and reason of this look at the level curves of (e.g.) $z=1$ & $z=0$

$z=1 \qquad 1 = f(x,y) = xy \Rightarrow \boxed{y = \frac{1}{x}}$

$z=0 \qquad 0 = f(x,y) = xy \Rightarrow \boxed{x=0 \text{ or } y=0}$



Notice that $\nabla f(x,y)$ seems to be orthogonal to the level curve through (x,y). This is not a coincidence.

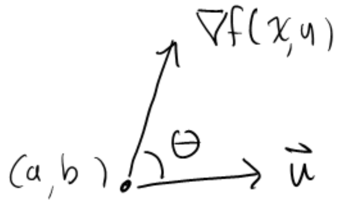
Gradients and Level Curves

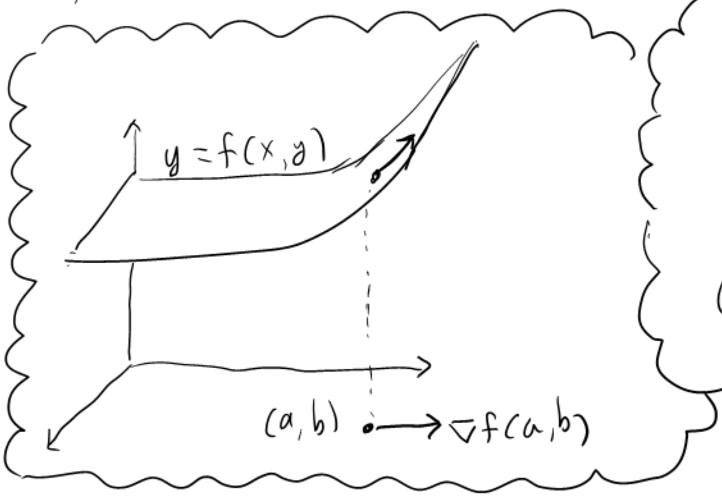
Consider the directional derivative of $f(x,y)$ in direction of $\vec{u}$. It gives the rate of change of $f(x,y)$ in the direction of $\vec{u}$:

$$\begin{pmatrix}\text{Rate of change} \\ \text{at } (a,b) \text{ of } f(x,y) \\ \text{in direction of } \vec{u}\end{pmatrix} = D_{\vec{u}} f(a,b) = \nabla f(a,b) \cdot \vec{u}$$

$$= |\nabla f(a,b)|\,|\vec{u}| \cos\theta$$

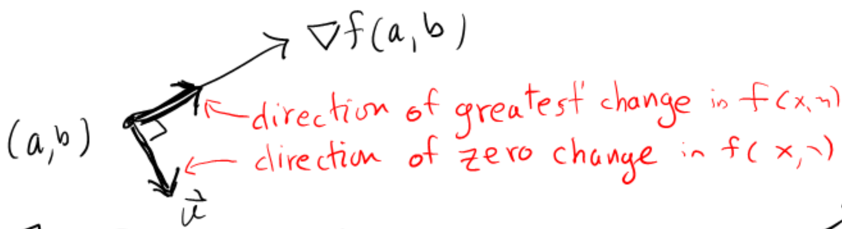
$$= |\nabla f(a,b)| \cos\theta$$





<u>zero</u> when $\cos\theta = 0$, i.e. $\theta = 90°$

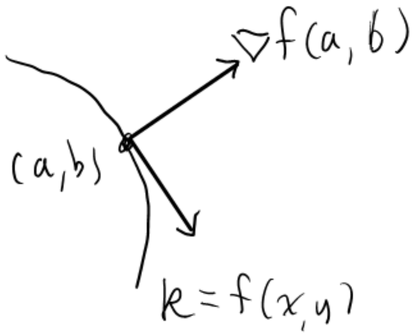
<u>greatest</u> when $\cos\theta = 1$, i.e. $\theta = 0°$

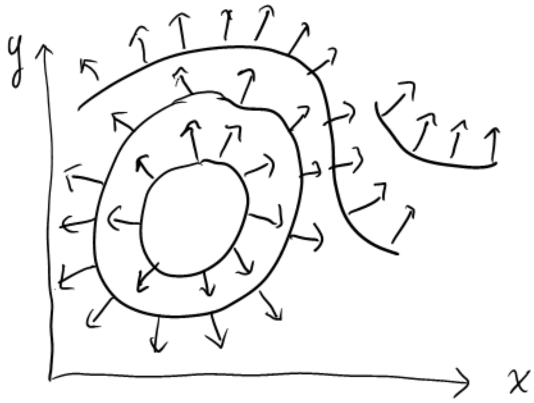


example

Conclusions:

- $\nabla f(a,b)$ points in the direction of the greatest rate of change of $f(x,y)$ at (a,b)
- $\nabla f(a,b)$ is perpendicular to the level curve through (a,b)





"Gradient field"
- family of vectors orthogonal to level curves of $y = f(x,y)$

Rules for the Gradient

① $\nabla(f \pm g) = \nabla f \pm \nabla g$
② $\nabla(kf) = k\nabla f$
③ $\nabla(fg) = (\nabla f)g + f(\nabla g)$
④ $\nabla\left(\frac{f}{g}\right) = \frac{(\nabla f)g - f(\nabla g)}{g^2}$

But you can almost always get by without these by first combining the functions then doing ∇.

Example Consider function $z = f(x,y) = x\cos(xy)$
You are standing at $(\frac{1}{2}, \pi)$ on xy-plane.
ⓐ In what direction should you move to create greatest rate of change in $f(x,y)$?
ⓑ What is that rate of change?
ⓒ What direction will effect zero change in $f(x,y)$.

$$\nabla f(x,y) = \left\langle \cos(xy) - x\sin(xy)y, \ -x\sin(xy)x \right\rangle$$

ⓐ $\nabla f(\frac{1}{2}, \pi) = \left\langle \cos(\frac{\pi}{2}) - \frac{1}{2}\sin(\frac{\pi}{2})\pi \ - \frac{1}{2}\sin(\frac{\pi}{2})\frac{1}{2} \right\rangle$

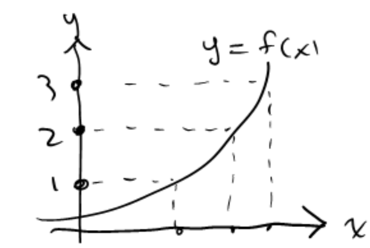
$= \left\langle 0 - \frac{\pi}{2}, -\frac{1}{4} \right\rangle = \boxed{\left\langle -\frac{\pi}{2}, -\frac{1}{4} \right\rangle}$

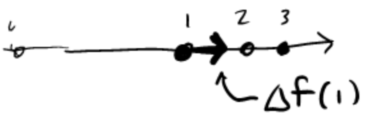
ⓑ $D_{\frac{\nabla f}{|\nabla f|}} f = \nabla f \cdot \frac{\nabla f}{|\nabla f|} = \frac{\nabla f \cdot \nabla f}{|\nabla f|} = \frac{|\nabla f|^2}{|\nabla f|} = |\nabla f|$

Answer: $|\nabla f| = \left|\left\langle -\frac{\pi}{2}, -\frac{1}{4} \right\rangle\right| = \sqrt{\frac{\pi^2}{4} + \frac{1}{16}} \approx 1.5905$ (z units per units in direction of ∇f)

ⓒ $\boxed{\left\langle \frac{1}{4}, -\frac{\pi}{2} \right\rangle}$ (because it's orthogonal to $\nabla f(\frac{1}{2}, \pi)$)

One Variable





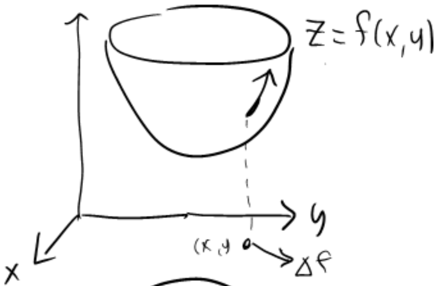
Level "points"

$\nabla f = \left\langle \frac{df}{dx} \right\rangle$

$= \left\langle f'(x) \right\rangle$

points in direction of greatest change in $f(x)$!

Two Variables



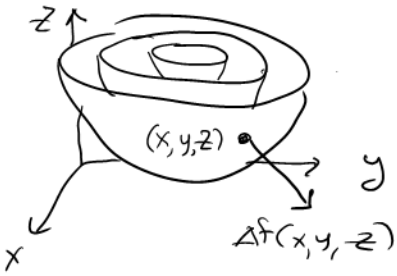
Level curves

$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$

points in direction of greatest change in $f(x,y)$.

Three variables

$z = f(x,y,z)$
Graph = ???



Level surfaces

$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$

Points in direction of greatest change in $f(x,y,z)$. at (x,y,z).
Orthogonal to level surface