Section 14.5 Directional Derivatives and Gradients

Recall

$$\frac{\partial f}{\partial x}(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h} = \left(\begin{array}{l}\text{rate of change of} \\ f(x,y) \text{ in the direction} \\ \text{of } \vec{i} = \langle 1,0\rangle\end{array}\right)$$

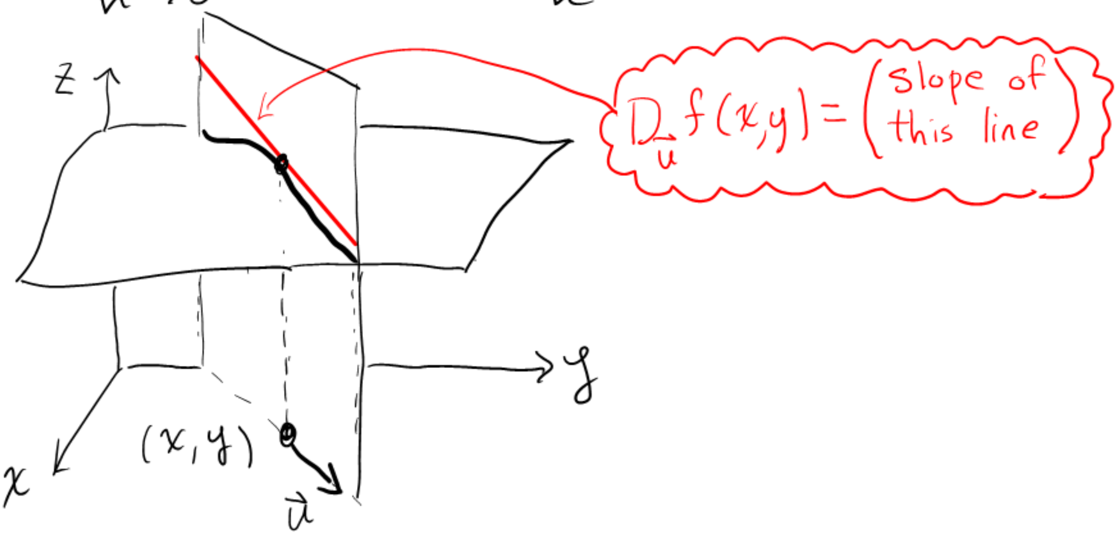
$$\frac{\partial f}{\partial y}(x,y) = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h} = \left(\begin{array}{l}\text{rate of change of} \\ f(x,y) \text{ in direction} \\ \text{of } \vec{j} = \langle 0,1\rangle\end{array}\right)$$

Now let $\vec{u} = \langle \vec{u}_1, \vec{u}_2 \rangle$ be a unit vector

$$D_{\vec{u}} f(x,y) = \lim_{h\to 0} \frac{f(x+hu_1, y+hu_2)-f(x,y)}{h} = \left(\begin{array}{l}\text{Rate of change of} \\ f(x,y) \text{ in direction} \\ \text{of } \vec{u} = \langle u_1, u_2\rangle\end{array}\right)$$

Definition Given a function $f(x,y)$ and unit vector $\vec{u} = \langle u_1, u_2 \rangle$, the directional derivative of f is the function

$$D_{\vec{u}} f(x,y) = \lim_{h\to 0} \frac{f(x+hu_1, y+hu_2)-f(x,y)}{h}$$



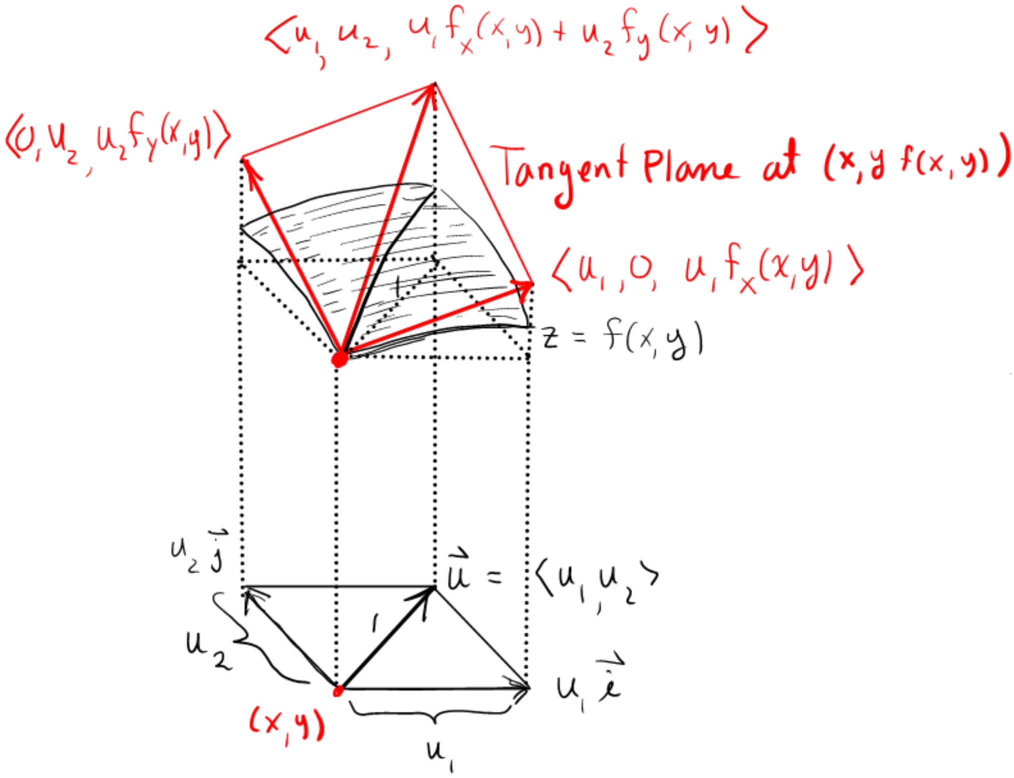
Note

$$\frac{\partial f}{\partial x}(x,y) = D_{\vec{i}} f(x,y)$$

$$\frac{\partial f}{\partial y}(x,y) = D_{\vec{j}} f(x,y)$$

So you can see it's easy to find a directional derivative in the special cases $\vec{u} = \vec{i}$ or $\vec{u} = \vec{j}$. Doing so for arbitrary $\vec{u}$ is almost as simple.

Heres a drawing of the siluation. A portion of the graph of $z = f(x,y)$ is shown and the plane tangent to the graph at $(x, y, f(z,y))$ is shown. The vector tangent to the surface over $\vec{u} = u_1\vec{i} + u_2\vec{j}$ is the sum of the vectors tangent to the surface over $u_1\vec{i}$ and $u_2\vec{j}$. Thus the tangent over $\vec{u}$ has run $|\vec{u}| = 1$ and rise $u_1 f_x(x,y) + u_2 f_y(x,y)$.



<u>Therefore</u>:

$$D_{\vec{u}} f(x,y) = f_x(x,y)\,u_1 + f_y(x,y)\,u_2$$
$$= \langle f_x(x,y), f_y(x,y) \rangle \cdot \langle u_1, u_2 \rangle$$
$$= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \cdot \vec{u}$$

<u>Example</u> Find the directional derivative of $f(x,y) = x^2y + y + 3$ at $P(5,2)$ in the direction of $\vec{u} = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$.

Solution $D_{\vec{u}} f(x,y) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \cdot \vec{u}$

$$= \langle 2xy, x^2+1 \rangle \cdot \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$$

$$= xy + \frac{\sqrt{3}(x^2+1)}{2}$$

Answer: $D_{\vec{u}} f(5,2) = 5\cdot 2 + \frac{\sqrt{3}(5^2+1)}{2} = \boxed{10 + 13\sqrt{3}}$

= (slope of tangent to surface over vector $\vec{u}$.)