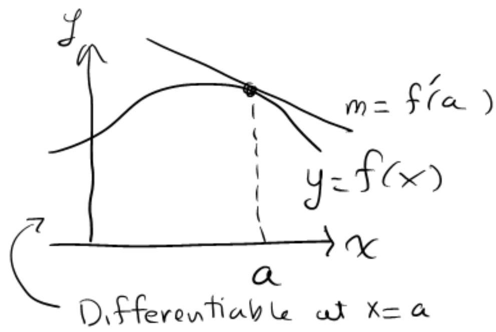
Section 14.3 Partial Derivatives (Revisited)

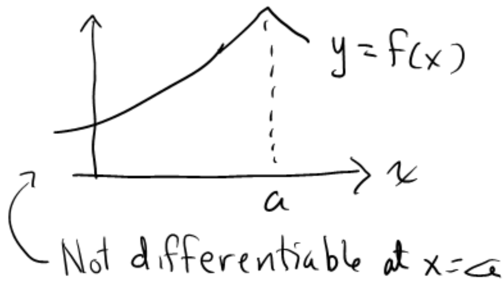
We now take up an important technical detail which we delayed earlier: Differentiability.



We say a function $y = f(x)$ is differentiable at $x = a$ if the following limit exists:

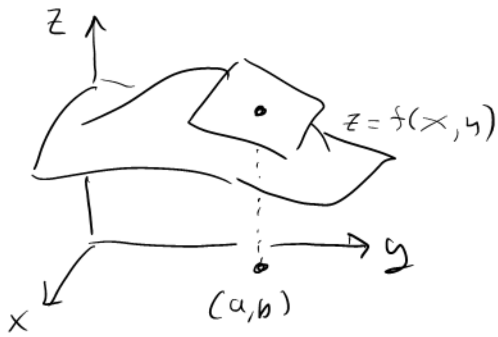
$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a).$$

Intuitively, this means that $y = f(x)$ has a tangent line at $(a, f(a))$, of slope $f'(a)$.

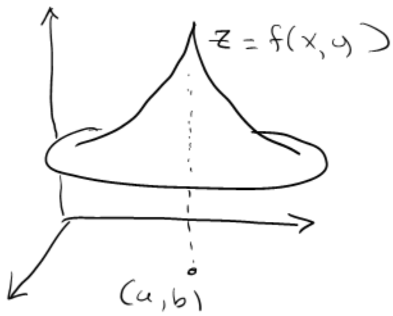


If the limit fails to exist, there is typically a cusp at $x = a$. In this case we say $f(x)$ is not differentiable at $x = a$

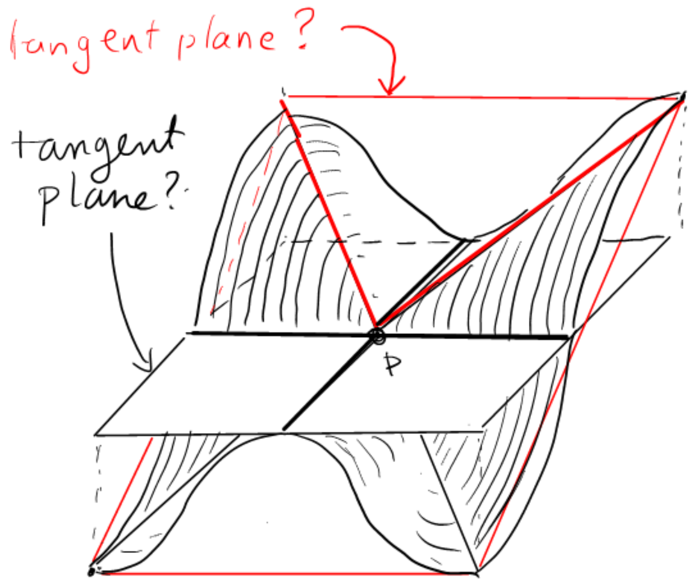
Now consider a function $z = f(x, y)$ of two variables. Intuitively, $f(x,y)$ being differentiable at (a,b) means that its graph at (a,b) is approximated by a tangent plane.



If, on the contrary, there is a cusp over (a,b) then there is no reasonable way to establish a tangent plane at $(a, b, f(a,b))$. We would say that the function is not differentiable at (a,b)



But the two-variable situation can be very subtle. Consider the function graphed on the right. At $(0,0)$ the directional derivatives exist in every direction. But is it approximated by a tangent plane at $(0,0)$? The picture illustrates two different reasonable choices for a tangent plane.



We need a careful mathematical investigation to sort all this out

For a point of departue, let's try to adapt the one variable case to two.

$f(x)$ is differentiable at a if

$$\lim_{dx \to 0} \frac{f(a+dx)-f(a)}{dx} = m$$

$\rightarrow$

$f(x,y)$ is differentiable at (a,b) if

$$\lim_{(dx,dy)\to(0,0)} \frac{f(a+dx,b+dy)-f(a,b)}{|\langle dx, dy\rangle|} = m$$

PROBLEM Tends not to exist. e.g. gives directional derivative for various approaches $(dx,dy)\to(0,0)$ Different values for different approaches.

rephrase to overcome problem

$$\lim_{dx\to 0} \frac{f(a+dx)-f(a)}{dx} = f'(a)$$

$$\lim_{dx\to 0} \left(\frac{f(a+dx)-f(a)}{dx} - f'(a)\right) = 0$$

$$\lim_{dx\to 0} \frac{f(a+dx)-f(a)-f'(a)\,dx}{dx} = 0$$

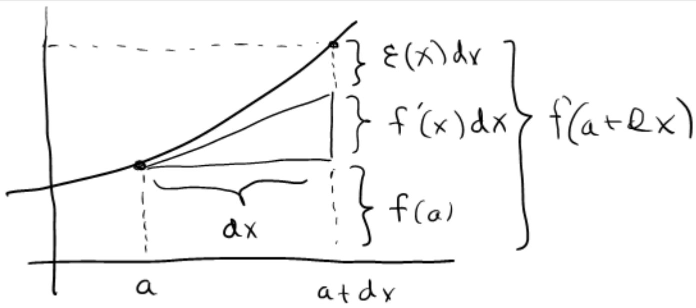
i.e. $\dfrac{f(a+dx)-f(a)-f'(a)\,dx}{dx} = \varepsilon(dx)$ and $\varepsilon(dx)\to 0$ as $dx\to 0$

i.e $f(a+dx)-f(a)-f'(a)\,dx = \varepsilon(dx)\,dx$ and $\varepsilon(dx)\to 0$ as $dx\to 0$

Definition $f(x)$ is differentiable at a if there is a number $f'(a)$ and a function $\varepsilon(dx)$ for which

$$f(a+dx) = f(a) + f'(a)\,dx + \varepsilon(dx)\,dx$$

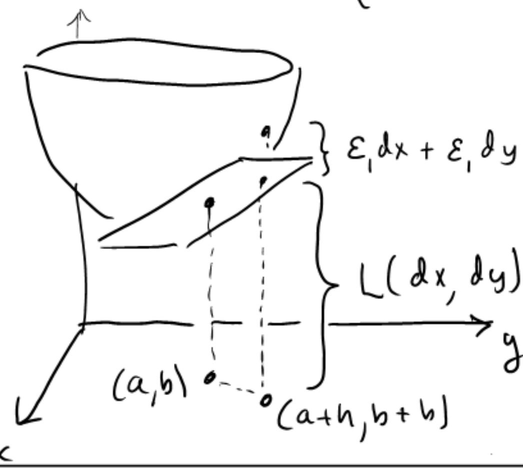
and $\varepsilon(dx)\to 0$ as $dx\to 0$



Definition $f(x,y)$ is differentiable at (a,b) if

$$f(a+dx, b+dy) = \underbrace{f(a,b) + f_x(a,b)\,dx + f_y(a,b)\,dy}_{\substack{\text{Linearization } L(dx,dy) \\ \text{- i.e. tangent plane}}} + \varepsilon_1\,dx + \varepsilon_2\,dy \quad \left\{\begin{array}{c}\varepsilon_1, \varepsilon_2 \to 0 \\ \text{as} \\ dx, dy \to 0\end{array}\right.$$

In essense, the definition says (in a very precise way) that The graph at (a,b) is approximated by a tangent plane.



For the most part in this course you can get by with ignoring issues of differentiability. But it does play an important role in the theoretical development of the material. You will come to appreciate that in more advanced courses such as MATH 407

For now we mention two theorems that hinge on differentiability,

<u>Theorem</u> $\left(\begin{array}{c} f_x \text{ and } f_y \text{ are continuous} \\ \text{on an open region } R \end{array}\right) \Longrightarrow \left(\begin{array}{c} f(x,y) \text{ is differentiable} \\ \text{on the region } R \end{array}\right)$

<u>Theorem</u> $\left(\begin{array}{c} f(x,y) \text{ is differentiable} \\ \text{at the point } (a, b) \end{array}\right) \Longrightarrow \left(\begin{array}{c} f(x,y) \text{ is continuous} \\ \text{at } (a, b) \end{array}\right)$