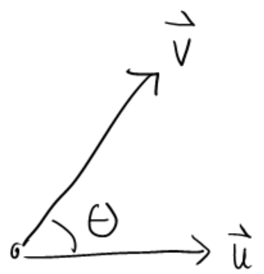
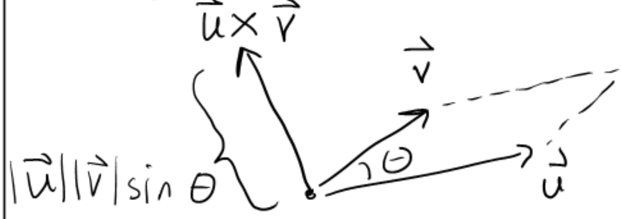
# Math 307 Section 12.5 Lines and Planes in Space

Before beginning, recall the following facts. Understanding them is the key to many computations involving the orientation of lines and planes in space.




$\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta$

$\theta = \cos^{-1}\left(\dfrac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}\right)$




Vector $\vec{u}\times\vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$. Its norm is $|\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin\theta$ = area of parallelogram with sides $\vec{u}, \vec{v}$.

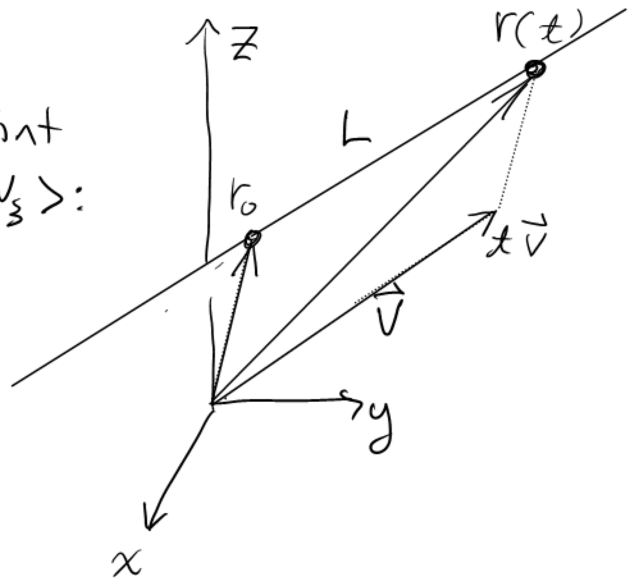
## Vector Equation for a Line in Space

Equation for line $L$ through point $\vec{r}_0 = \langle x_0, y_0, z_0\rangle$ parallel to $\vec{v} = \langle v_1, v_2, v_3\rangle$:

$$\vec{r}(t) = r_0 + t\vec{v}$$

$$= \langle x_0 + tv_1,\ y_0 + tv_2,\ z_0 + tv_3\rangle$$

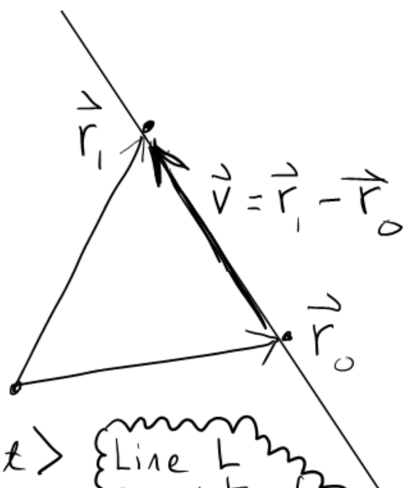
for $-\infty < t < \infty$.




Thus parametric form is $\begin{cases} x = x_0 + tv_1 \\ y = y_0 + tv_2 \\ z = z_0 + tv_3 \end{cases}$ for $-\infty \le t \le \infty$

**Example** Find equation of line through points $\vec{r}_0 = \langle 3, -1, 5\rangle$ and $\vec{r}_1 = \langle 2, 1, 2\rangle$

This line passes through $\vec{r}_0$ and has the same direction as $\vec{v} = \vec{r}_1 - \vec{r}_0$ (i.e. it's parallel to $\vec{v}$).




Thus equation is $\vec{r}(t) = \vec{r}_0 + t\vec{v}$

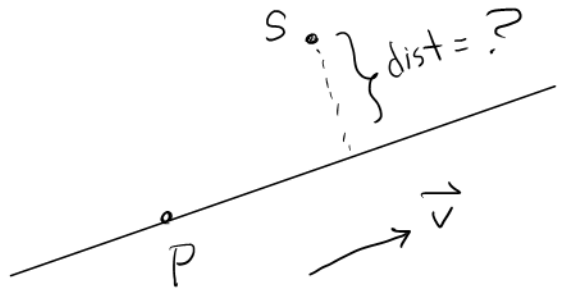
$= \langle 3, 1, 5\rangle + t\left(\langle 2, 1, 2\rangle - \langle 3, -1, 5\rangle\right)$

$= \langle 3, 1, 5\rangle + t\langle -1, 2, -3\rangle = \langle 3 - t, 1 + 2t, 5 - 3t\rangle$

$\boxed{r(t) = \langle 3 - t, 1 + 2t, 5 - 3t\rangle \text{ for } \infty < t < \infty}$

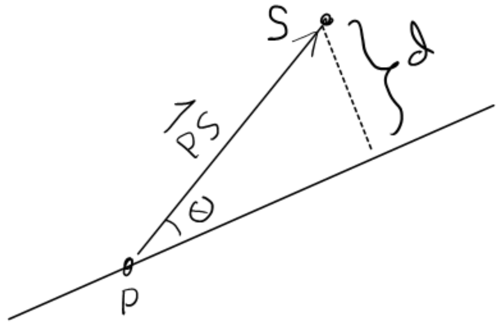
Line L consists of all these points

Sometimes you will have a line through a point P in the direction of a vector $\vec{v}$, and some other point S, and you will need to find the distance from S to the line, as indicated on the right.




By trigonometry, that distance is $d = |\vec{PS}| \sin\theta$

$$= \frac{|\vec{PS}||\vec{v}| \sin\theta}{|\vec{v}|} = \frac{|\vec{PS} \times \vec{v}|}{|\vec{v}|}$$




Conclusion:

Distance from a point S to a line through P parallel to $\vec{v}$:

$$\boxed{\frac{|\vec{PS} \times \vec{v}|}{|\vec{v}|}}$$

Note It's important to understand (not memorize) this formula, so if you forget it you can figure it out on the spot!

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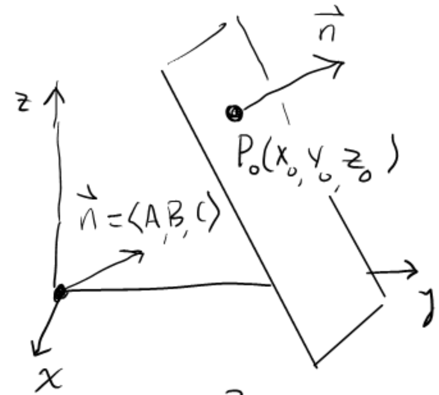
## Planes in Space:

We can specify a plane in space with two peices of information:

1. A point $P_0(x_0, y_0, z_0)$ on the plane
2. A vector $\vec{n} = \langle A, B, C \rangle$ normal to the plane




Given this information what is the equation of the plane?

Any point $P(x, y, z)$ on this plane satisfies

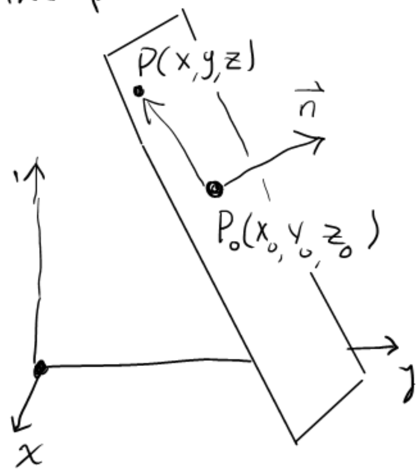
$$\vec{n} \cdot \vec{P_0P} = 0$$

$$\langle A, B, C \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

$$Ax + By + Cz = \underbrace{Ax_0 + By_0 + Cz_0}$$

$$Ax + By + Cz = D$$



## Conclusion

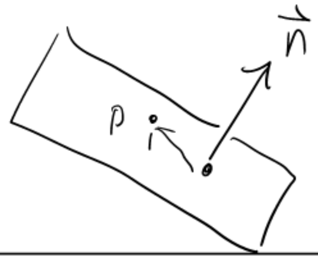
**Equation for a plane.**

The plane through point $P_0(x_0, y_0, z_0)$ and normal to $\vec{n} = \langle A, B, C\rangle$ is the set of all points $P(x, y, z)$ satisfying

$$\vec{n} \cdot \overrightarrow{P_0P} = 0,$$

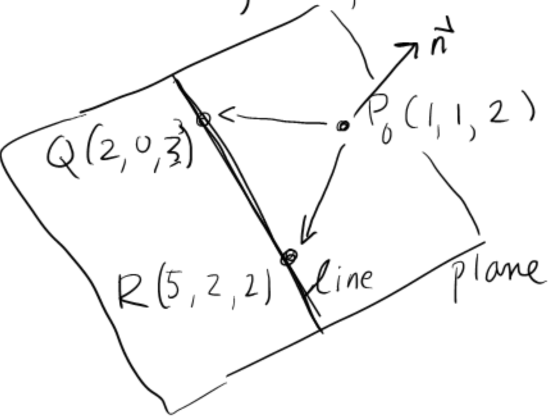
or $\quad Ax + By + Cz = D$

where $D = Ax_0 + By_0 + Cz_0$.




Typically, you may have to find the equation for a plane given some data that <u>does not</u> include the normal vector $\vec{v}$. In such a situation you'll need to compute the normal from the given data, usually using the cross product.

<u>Example</u> Find the equation of the plane containing the line $\langle 2+3t, 2t, 3-t\rangle$ and the point $P_0(1,1,2)$.




<u>Solution</u> We can get two specific points on the line as follows

$t = 0 \quad Q(2,0,3)$

$t = 1 \quad R(5,2,2)$

$t=0$ and $1$ chosen arbitrarily

$\overrightarrow{P_0Q} = \langle 1, -1, 1\rangle$

$\overrightarrow{P_0R} = \langle 4, 1, 0\rangle$

<u>Normal</u>: $\vec{n} = \overrightarrow{P_0Q} \times \overrightarrow{P_0R} = \begin{vmatrix} i & j & k \\ 1 & -1 & 1 \\ 4 & 1 & 0 \end{vmatrix} = \langle -1, 4, 5\rangle$

Point: $P_0(1,1,2)$

Equation $\quad -1\cdot x + 4y + 5z = -1\cdot 1 + 4\cdot 1 + 5\cdot 2$

$$\boxed{-x + 4y + 5z = 13} \leftarrow \text{Answer}$$

Plane consists of all points $P(x,y,z)$ satisfying this equation

$$\boxed{x - 4y - 5z = -13} \leftarrow$$

<u>Note</u> Multiplying both sides by any nonzero number (such as $-1$) yields an alternative (and equally valid) answer.

*Read other examples in text!!*

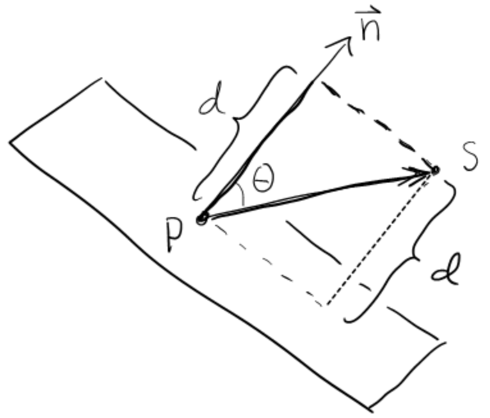
Using the dot and cross product, we can solve a variety of problems involving points, lines and planes.

**Distance $d$ from a point $S$ to a plane normal to $\vec{n}$, containing $P$.**

$$\cos\theta = \frac{\text{ADJ}}{\text{HYP}} = \frac{d}{|\vec{PS}|}$$

$$d = |\vec{PS}|\cos\theta$$

$$= \frac{|\vec{PS}||\vec{n}|\cos\theta}{|\vec{n}|} = \frac{\vec{PS}\cdot\vec{n}}{|\vec{n}|}$$



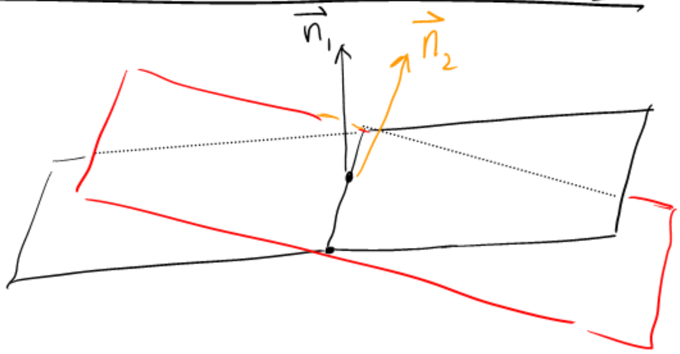

Thus distance $= \dfrac{|\vec{PS}\cdot\vec{n}|}{|\vec{n}|}$

(Absolute value necessary in case dot product is negative!)

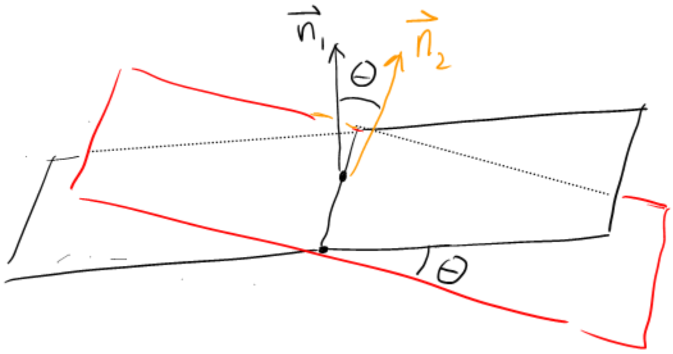
You are probably better off figuring out such problems from scratch (as above) rather than rembering the formula.

**Consider two planes with normal vectors $\vec{n}_1$ and $\vec{n}_2$.**

- Vector parallel to the line of intersection is $\vec{n}_1 \times \vec{n}_2$




- $\begin{pmatrix}\text{Angle } \theta \text{ between} \\ \text{the two planes}\end{pmatrix} = \begin{pmatrix}\text{angle formed} \\ \text{by normals}\end{pmatrix} = \cos^{-1}\left(\dfrac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}\right)$




Important: Hone your skills by reading examples in text and working exercises.

Avoid blind use of formulas. Draw a sketch and figure the problems out using trigonometry combined with the meanings of the dot & cross products.