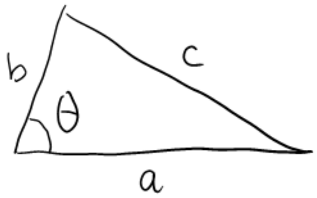
Recall

Law of Cosines



$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

where $0 \le \theta \le \pi$

Section 12.3 The Dot Product

The dot product of two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ gives significant information about their relationship to one another. The dot product of two vectors is a number

Definitions

In $\mathbb{R}^2$, the dot product of

$$\vec{u} = \langle u_1, u_2 \rangle$$

and $\vec{v} = \langle v_1, v_2 \rangle$

is $\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2$

In $\mathbb{R}^3$ the dot product of

$$\vec{u} = \langle u_1, u_2, u_3 \rangle$$

and $\vec{v} = \langle v_1, v_2, v_3 \rangle$

is $\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3$

Examples

- $\langle 2,5 \rangle \cdot \langle 3,2 \rangle = 2 \cdot 3 + 5 \cdot 2 = \boxed{16}$
- $\langle 1,1,-3 \rangle \cdot \langle 3,3,2 \rangle = 1 \cdot 3 + 1 \cdot 3 + (-3) \cdot 2 = \boxed{0}$
- $\langle 2,5 \rangle \cdot \langle 2,5 \rangle = 2^2 + 5^2 = \sqrt{2^2+5^2}^{\,2} = |\langle 2,5 \rangle|^2$

This last example suggests that the norm of a vector can be expressed as a dot product.

The dot product and norms

If $\vec{u} = \langle u_1, u_2 \rangle$ then $\vec{u} \cdot \vec{u} = u_1^2 + u_2^2 = |\vec{u}|^2$ so, $\boxed{|\vec{u}| = \sqrt{\vec{u} \cdot \vec{u}}}$

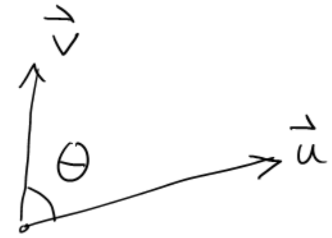
If $\vec{u} = \langle u_1, u_2, u_3 \rangle$ then $\vec{u} \cdot \vec{u} = u_1^2 + u_2^2 + u_3^2 = |\vec{u}|^2$, so $\boxed{|\vec{u}| = \sqrt{\vec{u} \cdot \vec{u}}}$

Properties

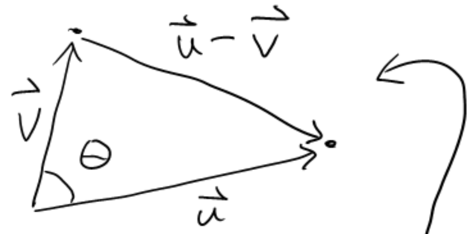
$$\left.\begin{array}{l} \vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u} \\ \vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w} \\ \vec{0} \cdot \vec{u} = 0 \\ (c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v}) = c\,\vec{u} \cdot \vec{v} \\ \vec{u} \cdot \vec{u} = |\vec{u}|^2 \end{array}\right\} \text{easy to check - see text}$$

Angle Between Two Vectors

The dot product can give the measure of the angle θ formed by vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$.



To see how, draw in $\vec{u}-\vec{v}$ and use dot product properties as follows:



$$|\vec{u}-\vec{v}|^2 = (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v})$$
$$= (\vec{u}-\vec{v})\cdot\vec{u} - (\vec{u}-\vec{v})\cdot\vec{v}$$
$$= \vec{u}\cdot\vec{u} - \vec{v}\cdot\vec{u} - \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v}$$
$$|\vec{u}-\vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2\,\vec{u}\cdot\vec{v}$$

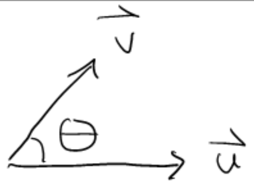
Law of Cosines:

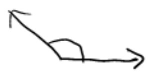
$$|\vec{u}-\vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos\theta$$

compare

Thus: $\boxed{\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta}$

IMPORTANT CONCLUSIONS



1. $\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta$
2. $\vec{u}\cdot\vec{v} > 0 \iff \cos\theta > 0 \iff \theta$ is accute 
3. $\vec{u}\cdot\vec{v} < 0 \iff \cos\theta < 0 \iff \theta$ is obtuse 
4. $\vec{u}\cdot v = 0 \iff \cos\theta = 0 \iff \theta = \pi/2$ 
   $\iff$ $\vec{u}$ and $\vec{v}$ are orthogonal (perpindicular)
5. $\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}$
6. $\theta = \cos^{-1}\left(\dfrac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}\right)$

Example What can we say about $\vec{u} = \langle 1,1,-3\rangle$ and $\vec{v} = \langle 3,3,2\rangle$?
Because $\vec{u}\cdot\vec{v} = 0$, these vectors are orthogonal in $\mathbb{R}^3$.

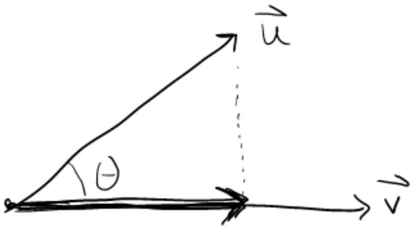
Example Find measure of angle formed by $\vec{u} = \langle 1,0,2\rangle$ and $\vec{v} = \langle 3,2,1\rangle$

$$\theta = \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}\right) = \cos^{-1}\left(\frac{5}{\sqrt{5}\sqrt{14}}\right) \approx 53.3^\circ \approx 0.929 \text{ radians}$$

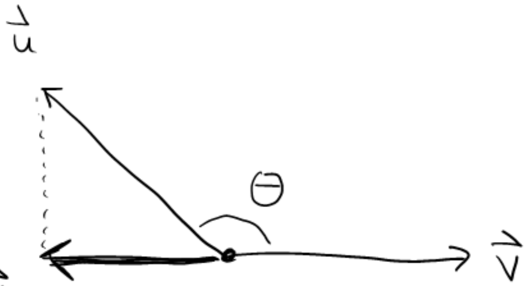
Example Angle between $\langle 0,1\rangle$ and $\langle 1,1\rangle$ is

$$\theta = \cos^{-1}\left(\frac{1}{1\cdot\sqrt{2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4} \text{ or } 45^\circ$$

Vector Projections

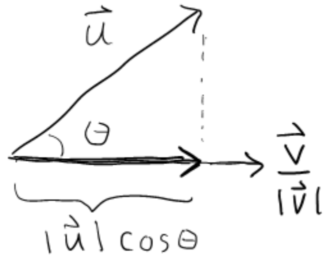


$\vec{w} = \text{proj}_{\vec{v}}\,\vec{u}$



$w = \text{proj}_{\vec{v}}\,\vec{u}$

The projection of $\vec{u}$ onto $\vec{v}$ is the vector $\vec{w} = \text{proj}_{\vec{v}}\,\vec{u}$ that is the projection of $\vec{u}$ straigth down to $\vec{v}$, as indicated.



It is the scalar multiple of the unit vector $\frac{\vec{v}}{|\vec{v}|}$ that has length $|\vec{u}|\cos\theta$

Thus
$$\text{proj}_{\vec{v}}\,\vec{u} = |u|\cos\theta\,\frac{\vec{v}}{|\vec{v}|}$$
$$= \frac{|\vec{u}||\vec{v}|\cos\theta}{|v|^2}\,\vec{v}$$
$$= \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\,\vec{v}$$

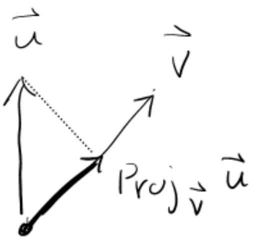
Conclusion

$$\boxed{\text{proj}_{\vec{v}}\,\vec{u} = \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\,\vec{v} = \underbrace{\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}}\,\frac{\vec{v}}{|\vec{v}|} = \underbrace{|u|\cos\theta}\,\frac{\vec{v}}{|\vec{v}|}}$$

Number $\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|} = |\vec{u}|\cos\theta$ is called the <u>scalar projection of $\vec{u}$ onto $\vec{v}$</u> because it's the number you multiply the unit vector $\frac{\vec{v}}{|\vec{v}|}$ by to get $\text{proj}_{\vec{v}}\,\vec{u}$

Example

$\vec{u} = \langle 0, 1\rangle$
$\vec{v} = \langle 1, 1\rangle$



$$\text{proj}_{\vec{v}}\,\vec{u} = \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\,\vec{v} = \frac{1}{2}\langle 1, 1\rangle$$
$$= \left\langle \frac{1}{2}, \frac{1}{2}\right\rangle$$