Meas. Sci. Technol. **11** (2000) 430–435. Printed in the UK PII: S0957-0233(00)09015-9

Estimation of non-statistical uncertainty using fuzzy-set theory

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Received 26 October 1999, in final form 18 January 2000, accepted for publication 3 February 2000

Abstract. A novel method using a fuzzy practicable interval to characterize non-statistical uncertainty in dynamic measurement is proposed. The method permits the uncertainty being estimated under the conditions that the number of measurements is very small and the probability distribution unknown. The feasibility of the method is validated by computer-simulation experiments.

Keywords: non-statistical uncertainty, estimation, dynamic measurement, fuzzy-set theory

1. Introduction

Measurement uncertainty shows the range within which the true value of a measurand is to be estimated [1]. It consists of statistical and non-statistical uncertainty [2–4]. For statistical uncertainty, some models have already been established [5]. If the probability-distribution density $p(x)$ of measured values obeys a normal distribution, its uncertainty can be estimated as $k\sigma_B$, where k is the confidence coefficient and σ_B is the standard deviation obtained by the Bessel method. If $p(x)$ is a non-normal distribution, a considerable estimation error is likely to occur if the same method is used [1, 4, 5].

The problem of non-normal distribution of measured values has been studied by several authors. Some typical distributions were examined by Sachs [6] using the square-graph method and the probability-paper method. Manonkian [7] and Pugachev [8] used the range-enlargement method. Hart [9] investigated the probability-treatment method.

Kubisa and Turzeniecka [10] evaluated a number of methods for approximating uncertainty in measurement when the convolution of the probability distribution of the error component as well as the output confidence level were known. Shu [11] suggests that the β distribution of the parameters α and β should be used for approximating the sample distribution; the confidence coefficients k_j and k_a can thus be obtained. The conceptual uncertainty [12] reveals three different ensembles of random values with distinct distribution laws. Bayesian theory and the maximum-entropy principle [13, 14] are based on statistics or the probability density.

The non-parametric statistical method [15] is of interest for non-Gaussian data under certain conditions. The method was further investigated by Lasserre *et al* [16] and Qin and Wang [17] in order to develop supplementary materials for GUM (Guide to the Expression of Uncertainty in Measurement) [3].

In dynamic measurements, estimation of the non-statistical uncertainty is very important, because not all the uncertainty can be estimated with the statistical method [18]. Furthermore, the case is typically far more complex than the statistical uncertainty. For example, in rocket-firing experiments and some destructive experiments, the number of measured values is rather small and the probability-distribution density of experiments of this kind might be unknown. In these cases, treating the problem by the available methods would be rather difficult.

To solve the above problems, the authors developed a new method using a fuzzy practicable interval to express the non-statistical uncertainty in dynamic measurement. The method allows the number of measurement values to be very small and the probability distribution unknown.

2. The fuzzy interval of measured data

By use of subordination functions, fuzzy mathematics [19] researches transition laws of a fuzzy entity changing from *true* to *false* or from *false* to *true*. In the measurement, the true value X_0 always exists uniquely and objectively. Therefore, we define a set A as

$$A = X_0 \tag{1}$$

The set A contains a single value X_0.

In set theory, for the measured values x_i, $i = 1, 2, \ldots, n$, the set A has the following characteristic function:

$$G_A(x) = \begin{cases} 1 & x_i \in A \\ 0 & x_i \notin A \end{cases} \tag{2}$$

where 1 represents true and 0 represents false. n is the number of measurements or the number of measurement samples.

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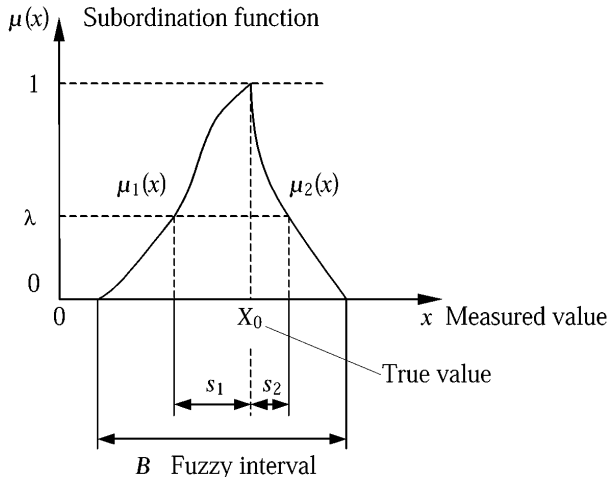

Figure 1. The subordination function and measured values.

In fuzzy-set theory, a transition can be considered to exist in the subordination of x_i in relation to A and the interval B of the transition can be described by the subordination function (figure 1):

$$\mu(x) = \begin{cases} \mu_1(x) & x_i \leqslant X_0 \\ \mu_2(x) & x_i \geqslant X_0 \end{cases} \qquad (3)$$

where $\mu_1(x) \in [0, 1]$ and $\mu_2(x) \in [0, 1]$. The function $\mu(x)$ describes how the measured value x_i accords with the set A. From figure 1, it can be seen that $\mu_1(x)$ increases while $\mu_2(x)$ decreases. $\lambda \in [0, 1]$. Two ranges s_1 and s_2 near X_0 on the x axis can be found. The interval of x subordinate to the set A is

$$U_{F_\lambda} = s_1 + s_2 \qquad (4)$$

where $\mu_{A_\lambda} = \lambda$. The value of $\mu(x)$ is the degree of subordination of x to the set A. In the measured value x_i, given $\lambda = \lambda^*$, then $U_{F_\lambda} = U_{F_{\lambda*}}$ is uniquely determined. This shows that the dispersal range of the measured value x_i is $U_{F_{\lambda*}}$ relative to the true value X_0 (figure 1). In figure 1, B is the fuzzy interval, λ^* is the optimum level and $U_{F_{\lambda*}}$ is the fuzzy practicable interval under the λ^* level. Then the characteristic function is given as

$$G_{A_\lambda}(x) = \begin{cases} 1\ \text{(true)} & \mu_A(x) \geqslant \lambda^* \\ 0\ \text{(false)} & \mu_A(x) < \lambda^*. \end{cases} \qquad (5)$$

Equation (5) shows that values of x in the interval $U_{F_{\lambda^*}}$ are usable, to be represented by 1 (true), whereas those outside the interval $U_{F_{\lambda^*}}$ are unusable, to be represented by 0 (false). On the basis of measurement theory, the uncertainty in measurement can be identified by $U_{F_{\lambda^*}}$.

3. The determination of λ^* and X_0

In terms of fuzzy-set theory, λ^* determines an entity's border from one extreme to another. In fact, λ^* can also be regarded as a fuzzy number and its fuzzy character reaches the peak when its value equals 0.5, both true and false. $\lambda \geqslant 0.5$ means that the most usable x is included in the set A. Therefore, in theory, λ^* can be determined to be 0.5. In practical data treatment, $\lambda^* = 0.4$–0.5 in general. If n is rather small, for example $n \leqslant 200$, $\lambda^* = 0.4$ can be adopted.

Generally speaking, the true value X_0 is unknown and it could be estimated by statistical analysis [20]. In figure 1, x_v, the value of x when $\mu(x) = 1$, is used to estimate X_0 as

$$X_0 \approx x|_{\mu(x)=1} = x_v. \qquad (6)$$

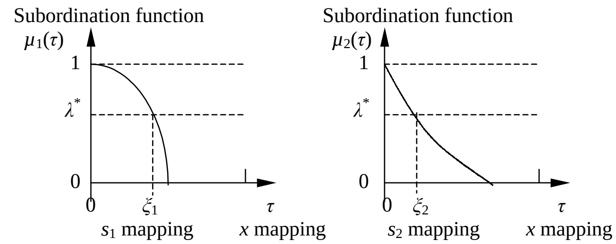

Figure 2. Subordination functions and mapping parameters.

4. Parameter mapping

The subordination function in fuzzy mathematics can be identified by a probability-distribution-density function in error theory. If $p = p(x)$ is known, the linear transformation

$$\mu(x) = (p(x) - p_{min})/(p_{max} - p_{min}) \qquad p_{min} \neq p_{max} \qquad (7)$$

maps p into the interval $[0, 1]$ and hence $\mu(x)$ can be obtained. In equation (7), the suffix min and the suffix max stand for the minimum and maximum values, respectively. Equations (6) and (7) conclude that x_v is in correspondence with p_{max}, which is the maximum value of the probability-distribution-density function.

Since x_i is already regarded as a fuzzy number, it lies in the interval $[0, 1]$. Therefore, the linear transformations

$$\eta_v = (x_v - x_{min})/(x_{max} - x_{min}) \qquad (8)$$

$$\eta(x) = (x - x_{min})/(x_{max} - x_{min}) \qquad (9)$$

$$\tau = \tau(x) = |\eta(x) - \eta_v| = |x - x_v|/(x_{max} - x_{min}) \qquad (10)$$

map x into the interval $[0, 1]$ and the measured value written as the fuzzy number $\tau(x)$ is obtained, where x_v can be expressed as $\tau_v = 0$.

In the interval $[0, 1]$, ψ_{F_λ} stands for U_{F_λ} and ξ_1 and ξ_2 stand for s_1 and s_2, respectively. Then equation (4) can be written as

$$\begin{aligned} U_{F_\lambda} &= s_1 + s_2 = (|x - x_v|_{\mu_1(x)=\lambda}) + (|x - x_v|_{\mu_2(x)=\lambda}) \\ &= (x_{max} - x_{min})(\tau|_{\mu_1(\tau)=\lambda}) + (x_{max} - x_{min})(\tau|_{\mu_2(\tau)=\lambda}) \\ &= (\tau|_{\mu_1(\tau)=\lambda} + \tau|_{\mu_2(\tau)=\lambda})(x_{max} - x_{min}) \\ &= (\xi_1 + \xi_2)(x_{max} - x_{min}) = \psi_{F_\lambda}(x_{max} - x_{min}) \end{aligned} \qquad (11)$$

$$\psi_{F_\lambda} = \xi_1 + \xi_2. \qquad (12)$$

On the basis of the above, figure 1 can be extended further, as shown in figure 2.

If the discrete values $\mu_{1j}(\tau_j)$ and $\mu_{2j}(\tau_j)$, $j = 1, 2, \ldots,$ are known, $\mu_1(\tau)$ and $\mu_2(\tau)$ can be obtained using the following method. Define the maximum norm

$$\|r\|_\infty = \max|r_j| \qquad j = 1, 2, \ldots. \qquad (13)$$

Use the polynomials

$$f_1 = f_1(\tau) = 1 + \sum_{l=1}^{L} a_l \tau^l \tag{14}$$

$$f_2 = f_2(\tau) = 1 + \sum_{l=1}^{L} b_l \tau^l \tag{15}$$

to approximate discrete values $\mu_{1j}(\tau_j)$ and $\mu_{2j}(\tau_j)$, respectively. We obtain

$$\mu_1(\tau) = f_1(\tau) \tag{16}$$

$$\mu_2(\tau) = f_2(\tau). \tag{17}$$

Suppose that

$$r_{1j} = f_1(\tau_j) - \mu_{1j}(\tau_j) \qquad j = 1, 2, \ldots, v \tag{18}$$

$$r_{2j} = f_2(\tau_j) - \mu_{2j}(\tau_j) \qquad j = v, v+1, \ldots. \tag{19}$$

Select $a_l = a_l^*$ that satisfies

$$\min \|r_1\|_\infty. \tag{20}$$

Select $b_l = b_l^*$ that satisfies

$$\min \|r_2\|_\infty. \tag{21}$$

Then the unknown coefficients a_l and b_l can be obtained. In equations (14) and (15), L, the degree of polynomials, is typically equal to 3 or 4.

The restraint conditions in equations (20) and (21) can be expressed as

$$f_1' = \mathrm{d}f_1/\mathrm{d}\tau \leqslant 0 \tag{22}$$

$$f_2' = \mathrm{d}f_2/\mathrm{d}\tau \leqslant 0. \tag{23}$$

This reveals a steadily decreasing characteristic regarding the subordination functions. This method is called the maximum-norm method and the error of the approximation is less than that of the least-squares method. ξ_1 and ξ_2 can be obtained from the following expressions:

$$\min|\mu_1(\tau) - \lambda^*|_{\tau=\xi_1} \tag{24}$$

$$\min|\mu_2(\tau) - \lambda^*|_{\tau=\xi_2}. \tag{25}$$

The right-square-graph estimation method is described first. When n is rather large, for example $n \geqslant 30$, the measured values can be divided into q groups. In each group, the median is d_j and the frequency is m_j.

Let the median of the group having the largest frequency be x_v and the number of the group be v. If there are t repeated frequencies, x_v and v can be determined by the mean method. Then

$$p_{1j}(x_j) = m_j \qquad j = 1, 2, \ldots, v \tag{26}$$

$$p_{2j}(x_j) = m_j \qquad j = v, v+1, \ldots, q+1. \tag{27}$$

Another approach is to use the linear estimation method. Arrange x_i from small to large to form a new order as

$$x_i^* \leqslant x_{i+1}^* \qquad i = 1, 2, \ldots, n-1. \tag{28}$$

Define

$$\Delta_i = x_{i+1}^* - x^* \geqslant 0. \tag{29}$$

In general, the smaller Δ_i the thicker the distribution of the measured values. Conversely, it will be thinner. Δ_i and x_i are related to the distribution density. We use the linear functions

$$m_j = 1 - (\Delta_j - \Delta_{min})/\Delta_{max} \qquad \Delta_{max} = \max \Delta_j$$

$$\Delta_{min} = \min \Delta_j \qquad j = 1, 2, \ldots, n-1 \tag{30}$$

to approximate the probability-distribution-density function. Similarly to the right-square-graph estimation method, x_v and v can be obtained. Therefore,

$$p_{1j}(x_j^*) = m_j \qquad j = 1, 2, \ldots, v \tag{31}$$

$$p_{2j}(x_j^*) = m_j \qquad j = v, v+1, \ldots, n. \tag{32}$$

$\mu_{1j}(\tau_j)$ and $\mu_{2j}(\tau_j)$ can be obtained using equations (7)–(23).

5. The estimation algorithm

The algorithm of the method is based on the optimal theory. The target functions are equations (20), (21), (24) and (25). The constraint conditions are equations (22) and (23). The optimal method used is SUMT [21]. The algorithm is summarized as follows.

(i) Sample x_i $\quad i = 1, 2, \ldots, n$.
(ii) Obtain a new order x_i^* according to equation (28).
(iii) Compute $p_{1j}(x_j^*) = m_j$ $(j = 1, 2, \ldots, v)$ and $p_{2j}(x_j^*) = m_j$ $(j = v, v+1, \ldots, n)$ according to equations (29)–(32) after obtaining v and x_v.
(iv) Compute $\eta(x)$ and τ according to equations (9) and (10) after obtaining $\mu_{1j}(\tau_j)$ $(j = 1, 2, \ldots, v)$ and $\mu_{2j}(\tau_j)$ $(j = v, v+1, \ldots, n)$ from equation (7).
(v) Establish the models for f_1 and f_2 from equations (14), (15) and (18)–(21) under the condition of equations (22) and (23).
(vi) Obtain the subordination functions $\mu_1(\tau)$ and $\mu_2(\tau)$ according to equations (16) and (17).
(vii) Obtain $U_{F_{\lambda^*}}$ according to equation (11) after computing ξ_1 and ξ_2 under the level $\lambda = \lambda^*$ from equations (24) and (25).

6. Case studies

The estimation error of the proposed method is to be examined. The simulated data are used to analyse the uncertainty in the estimation error and the normal distribution, Rayleigh distribution, triangle distribution and uniform distribution are considered in the investigation.

The simulated values x_i of the four distributions are generated by the computational method. The values of $U_{F_{\lambda^*}}$ for the four distributions are obtained. Suppose that the true value of the uncertainty is U; then the relative error in the estimation for U can be defined as

$$\Delta U = |U_{F_{\lambda^*}} - U|/U. \tag{33}$$

In the simulation, $\lambda^* = 0.4$ to take a small sample ($4 \leqslant n \leqslant 50$). $\mu(x)$ can be estimated using the proposed estimation method.

Table 1. Results of the simulation and case studies ($n = 6$–10).

		Distribution			
		Normal	Rayleigh	Triangle	Uniform
True value of uncertainty		0.6	1.723 01	1.0	1.0
Proposed method	Estimated uncertainty	0.574 453 0.333 788	1.842 456	1.069 633 0.9246 79	0.958 79
	Relative error (%)	4.26 44.37	6.932	6.96 7.53	4.121
Bessel method	Estimated uncertainty	0.690 54 0.375 94	2.097 77	1.225 55 1.416 87	1.773 81
	Relative error (%)	15.09 37.34	21.75	22.56 41.69	77.38

Table 2. Results of the simulation and case studies ($n = 4$).

		Distribution			
		Normal	Rayleigh	Triangle	Uniform
True value of uncertainty		0.6	1.723 01	1.0	1.0
Proposed method	Estimated uncertainty	0.480 01	1.410 93	0.943 80	0.829 55
	Relative error (%)	20	18.11	5.62	17.05

6.1. The normal distribution

The first set of simulated values x_i ($n = 10$) is as follows:

$$\begin{array}{ccccc} 50.021\,70 & 50.042\,82 & 50.066\,41 & 49.922\,50 & 49.711\,10 \\ 50.002\,18 & 50.125\,76 & 50.032\,76 & 50.021\,37 & 49.912\,43. \end{array}$$

The true value U in the simulation is $U = 6\sigma = 0.6$.

The uncertainty estimated using the proposed method is $U_{F_{\lambda *}} = U_{F_{0.4}} = 0.574\,453$. The relative error of the estimation is

$$\Delta U = |0.574\,453 - 0.6|/0.6 = 4.26\%.$$

By Bessel estimation, the uncertainty is $6\sigma_B = 0.690\,54$. The relative error of estimation is 15.09%.

The second set of simulated values x_i ($n = 10$) is as follows:

$$\begin{array}{ccccc} 49.935\,76 & 49.991\,86 & 49.980\,47 & 49.971\,54 & 49.995\,12 \\ 49.851\,21 & 49.939\,83 & 49.960\,95 & 50.084\,53 & 50.040\,63. \end{array}$$

The true value U in the simulation is $U = 6\sigma = 0.6$.

The uncertainty estimated using the proposed method is $U_{F_{\lambda *}} = U_{F_{0.4}} = 0.333\,788$. The relative error of the estimation is 44.37%.

By the Bessel estimation, the uncertainty is $6\sigma_B = 0.375\,94$. The relative error of the estimation is 37.34%.

6.2. The Rayleigh distribution

The simulated values x_i ($n = 10$) are as follows:

$$\begin{array}{ccccc} 1.6201 & 2.138 & 0.8592 & 1.541\,93 & 1.209\,55 \\ 1.743\,25 & 1.189\,94 & 1.526\,84 & 1.522 & 1.659\,53. \end{array}$$

The true value U in the simulation is $U = 2 \times 2.636\sigma_R = 1.723\,01$.

The uncertainty estimated using the proposed method is 1.842 456 and the relative error of the estimation is only 6.932%.

By the Bessel estimation, the uncertainty is $6\sigma_B = 2.097\,77$ and the relative error of the estimation is 21.75%.

6.3. The triangle distribution

The first set of simulated values x_i ($n = 10$) is as follows:

$$\begin{array}{ccccc} 5.544\,24 & 5.625\,905 & 5.7059 & 5.691\,365 & 5.801\,535 \\ 5.455\,76 & 5.374\,095 & 5.2941 & 5.308\,635 & 5.198\,465. \end{array}$$

The interval of the simulation is $[5, 6]$. The true value of the uncertainty is $U = 6 - 5 = 1$.

The uncertainty estimated using the proposed method is 1.069 633 and the relative error of the estimation is 6.96%.

By the Bessel estimation, the uncertainty is $6\sigma_B = 1.225\,55$ and the relative error of the estimation is 22.56%.

The second set of simulated values x_i ($n = 8$) is as follows:

$$\begin{array}{cccc} 5.561\,17 & 5.599\,51 & 5.920\,055 & 5.583\,92 \\ 5.4883 & 5.400\,49 & 5.079\,945 & 5.416\,08. \end{array}$$

The interval is $[5, 6]$ and the true value of the uncertainty is $U = 6 - 5 = 1$.

The uncertainty estimated using the proposed method is 0.924 679 and the relative error of the estimation is 7.53%.

By the Bessel estimation, the uncertainty is $6\sigma_B = 1.416\,87$ and the relative error of the estimation is 41.69%.

6.4. The uniform distribution

The simulated values x_i ($n = 7$) are as follows:

$$\begin{array}{cccc} & 5.050\,303 & 5.573\,13 & 5.338\,94 \\ 5.000\,67 & 5.3255 & 5.858\,51 & 5.424\,44. \end{array}$$

Table 3. Results of the simulation and case studies ($n = 20$).

		Distribution			
		Normal	Rayleigh	Triangle	Uniform
True value of uncertainty		0.6	1.72301	1.0	1.0
Proposed method	Estimated uncertainty	0.657	1.692 18	0.922 21	0.966 04
	Relative error (%)	9.5	1.789	7.779	3.396

Table 4. Results of the simulation and case studies ($n = 50$).

		Distribution			
		Normal	Rayleigh	Triangle	Uniform
True value of uncertainty		0.6	1.72301	1.0	1.0
Proposed method	Estimated uncertainty	0.626	1.812 50	1.101 73	0.932663
	Relative error (%)	4.333	5.194	10.173	6.734

The interval is [5, 6] and the true value of the uncertainty is $U = 6 - 5 = 1$.

The uncertainty estimated using the proposed method is 0.958 79 and the relative error of the estimation is 4.121%.

By the Bessel estimation, the uncertainty is $6\sigma_B = 1.773\,81$ and the relative error of the estimation is 77.38%.

6.5. Summary

The results of the above four cases are summarized in table 1. To further demonstrate the effectiveness of the proposed method, the sampling number n is varied for the above cases and the results are shown in table 2–4.

7. Discussion

It can be seen that the proposed method can be used for the normal, Rayleigh, triangle and uniform distributions both for large n and for small n. According to the results in tables 1–4, the proposed method can be used for small samples, for example as few as four elements. The results for samples of six or more elements are rather good.

The simulation results show that, when measured values conform to the normal distribution, the relative error of the estimation using the proposed method is very small. The confidence level has been computed and it is up to 99.73%. The relative error of the estimation using the Bessel method is very large under the condition of a small sampling number.

When measured values conform to non-normal distributions, such as the Rayleigh distribution, the triangle distribution and the uniform distribution, the results obtained using the proposed method are near to the true values of the uncertainty and the relative errors of the estimation are very small. However, using the Bessel method, the errors are fairly large. The main reason for the large error is that the Bessel method can be applied only under the condition of the distribution being normal.

This shows that the proposed method can be applied to various conditions, in particular, the conditions of non-normal distributions. Thus, when the distributions of measured values are unknown, results that are very near to the true values can be obtained using the proposed method. The reason is that the estimation method developed can automatically recognize the subordination function $\mu(x)$ according to the discrete characteristics, which may be unknown prior to the measurement.

The uncertainty under λ^* of measured values can be directly obtained using this method. With the new method, it is no longer necessary to estimate the standard deviation and the confidence coefficients. For systems of small samples and unknown distributions, the proposed method is more suitable.

8. Conclusions

The fuzzy practicable interval U_{F_λ} can be used as the estimation parameter for the uncertainty of measured values. Its optimal level is 0.4–0.5. U_{F_λ} is relative to x_v, the value of x when the degree of subordination equals 1. Therefore x_v can be regarded as an estimation parameter of the true value.

The subordination function can be worked out from measured values through the right-square graph and the proposed estimation method without the need to know the probability-distribution density. It can be expressed as a polynomial under the condition of minimizing the maximum norm.

Using the fuzzy practicable interval U_{F_λ} to estimate the uncertainty of measured values is characterized by allowing the distribution of measured values to be unknown and the number of samples to be very small. The proposed method can be applied to the estimation of non-statistical uncertainty in dynamic measurement.

Acknowledgment

This project is supported by the National Science Foundation of the People's Republic of China (grant 59 805 007).

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