Periodicity and Convergence in $x_{n+1} = |x_n - x_{n-1}|$

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Abstract. Each solution $\{x_n\}$ of the equation in the title is either eventually periodic with period 3 or else, it converges to zero – which case occurs depends on whether the *ratio* of the initial values of $\{x_n\}$ is rational or irrational. Further, the sequence of ratios $\{x_n/x_{n-1}\}$ satisfies a first order difference equation that has periodic orbits of all integer periods *except* 3. p-cycles for each $p \neq 3$ are explicitly determined in terms of the Fibonacci numbers. In spite of the non-existence of period 3, the unique positive fixed point of the first order equation is shown to be a snap-back repeller so the irrational ratios behave chaotically.

1 Introduction

Consider the second-order difference equation

$$x_{n+1} = |x_n - x_{n-1}|, \qquad n = 0, 1, 2, \ldots \tag{1}$$

For $n = 0$, we may assume that the initial values x_{-1}, x_0 are non-negative and for non-triviality, at least one is positive. In [5], the related equation

$$x_{n+1} = cx_n + a|x_n - x_{n-1}| \tag{2}$$

is discussed as a member of a more general class, and in particular it is shown that for $0 \leq c < 1/2$ and $c < a < 1 - c$ every (non-negative) solution of (2) converges to zero in a non-monotonic fashion. Also, for a range of a, c values within the open interval (0,1), it is shown in [5] that the ratios x_n/x_{n-1} oscillate in a chaotic manner thereby causing highly irregular, off-equilibrium oscillations in the converging solutions of (2).

The purpose of this note is to give a complete characterization of the asymptotic behaviors of the solutions of Equation (1), which may be obtained from (2) by setting $a = 1$ and $c = 0$. In particular, the solutions of (1) are seen to behave very differently than the solutions of (2) with $c = 0$ and $0 < a < 1$. Books such as [2] and [6] contain all of the basic background that may be needed for this paper.

Dividing (1) on both sides by x_n gives

$$\frac{x_{n+1}}{x_n} = \left|1 - \frac{x_{n-1}}{x_n}\right|$$

which can be written as

$$r_{n+1} = \left|\frac{1}{r_n} - 1\right|, \quad n = 0, 1, 2, \ldots \tag{3}$$

if we define $r_n = x_n/x_{n-1}$ for every $n \geq 0$. We may think of (3) as the recursion $r_{n+1} = \phi(r_n)$ where ϕ is the piecewise smooth mapping

$$\phi(r) = \left|\frac{1}{r} - 1\right|, \quad r > 0.$$

In this format, solutions $\{r_n\}$ of (3) can be written as $r_n = \phi^n(r_0)$ for $n \geq 1$. Since ϕ is not defined at $r = 0$, the iteration process for ϕ stops at step k if $\phi^k(r) = 0$ for some $r > 0$. For example, $\phi(1) = 0$ so $k = 1$ when $r = 1$. Such values of r are generally determined by iterating ϕ backward from 0 to get

$$C = \cup_{i=0}^{\infty} \phi^{-i}(0) = \{r > 0 : \phi^i(r) = 0 \text{ for some positive integer } i\} \cup \{0\}.$$

In the above definition, we interpret ϕ^0 as the identity mapping. The next result establishes a basic property of the set C with respect to the solutions of (1).

Lemma 1. *If $\{x_n\}$ is a solution of (1) with $x_0/x_{-1} \in C$, then $\{x_n\}$ eventually has period 3.*

Proof. By assumption $r_0 = x_0/x_{-1} \in C$; therefore, $x_{-1} \neq 0$ and there is $k \geq 0$ such that $r_k = \phi^k(r_0) = 0$ for some least integer k. Hence, $x_k = 0$ and it readily follows that

$$\{x_n\} = \{x_{-1}, x_0, \ldots x_{k-1}, 0, x_{k-1}, x_{k-1}, 0, x_{k-1}, x_{k-1}, 0, \ldots\}.$$

In the sequel, it is convenient to use the following "halves" of ϕ:

$$\phi_1(r) = \frac{1}{r} - 1, \ 0 < r \leq 1, \quad \phi_2(r) = 1 - \frac{1}{r}, \ r \geq 1.$$

Notice that both ϕ_1 and ϕ_2 are one-to-one maps and their inverses are easily computed as

$$\phi_1^{-1}(r) = \frac{1}{1+r}, \ r \geq 0, \quad \phi_2^{-1}(r) = \frac{1}{1-r}, \ 0 \leq r < 1.$$

The mapping ϕ has a unique fixed point

$$\bar{r} = \frac{\sqrt{5} - 1}{2}$$

which is the same as the unique fixed point for ϕ_1 because ϕ_2 does not intersect the 45-degree line. Next, we define the set

$$D = \cup_{i=0}^{\infty} \phi^{-i}(\bar{r}) = \{r > 0 : \phi^i(r) = \bar{r} \text{ for some positive integer } i\}$$

Note that $\bar{r} = \phi^0(\bar{r}) \in D$ and that $D \cap C$ is empty. The next result establishes a basic property of the set D based on the fact that $\bar{r} < 1$; the simple proof is omitted.

Lemma 2. *If $\{x_n\}$ is a solution of (1) with $x_0/x_{-1} \in D$, then $\{x_n\}$ is eventually decreasing monotonically to zero.*

2 The asymptotic dichotomy

Since ϕ is a rational form, we see that $C \subset \mathbb{Q}^+$, where $\mathbb{Q}^+$ is the set of all non-negative rational numbers. However, $D \cap \mathbb{Q}^+$ is empty. Therefore, by Lemmas 1 and 2, period-3 solutions of (1) may exist when the initial values x_0, x_{-1} are rational, whereas solutions that converge to zero can occur when the initial values are irrational. Theorem 1 below shows that this dichotomy is descriptive of *all* solutions of (1). We need one more lemma before stating the theorem.

Lemma 3. *Let $\{x_n\}$ be a solution of (1). If $x_k > x_{k-1}$ for some $k \geq 0$, then $x_n < x_k$ for all $n > k$.*

Proof. Under the given hypotheses we have that $x_{k+1} = x_k - x_{k-1} < x_k$. Therefore, $x_{k+2} = x_k - x_{k+1} < x_k$, and thus, $x_{k+3} \leq \max\{x_{k+1}, x_{k+2}\} < x_k$. The last step by induction extends to $n > k+3$ and completes the proof.

Theorem 1. (a) *If $x_0/x_{-1} \notin \mathbb{Q}^+$ then the corresponding solution $\{x_n\}$ of (1) converges to zero.*

(b) *If $x_0/x_{-1} \notin \mathbb{Q}^+ \cup D$ then the solution $\{x_n\}$ converges to zero but it is not eventually monotonic.*

(c) *$C = \mathbb{Q}^+$; thus if $x_0/x_{-1} \in \mathbb{Q}^+$ then the corresponding solution $\{x_n\}$ of (1) has period 3 eventually.*

Proof. (a) Since $r_0 = x_0/x_{-1} \notin C$, it follows that $r_n \neq 0, 1$ for all n. This implies that $x_n \neq 0, x_{n-1}$ for all n. Therefore, either $x_n < x_{n-1}$ for all n in which case x_n converges to zero monotonically, or there is $k_1 \geq 0$ such that $x_{k_1} > x_{k_1-1} > 0$. In the latter case, Lemma 3 implies that $x_n < x_{k_1}$ for all $n > k_1$. If the sequence $\{x_n\}$ is not eventually decreasing, then there is an increasing sequence k_i of positive integers such that

$$x_{k_1} > x_{k_2} > \cdots > x_{k_i} > \cdots$$

and for $i = 1, 2, 3, \ldots$

$$x_n < x_{k_i} \quad \text{if} \quad k_i < n \leq k_{i+1}.$$

These facts imply that $x_n \to 0$ as $n \to \infty$.

(b) Convergence follows from Part (a). If $\{x_n\}$ is eventually monotonic, then there is $k \geq 0$ such that $x_n < x_{n-1}$ or equivalently, $r_n < 1$ for all $n \geq k$. We show that this leads to a contradiction. Since $r_0 \notin D$, it follows that $r_n \neq \bar{r}$ for all n. Note that for $r \in (1/2, \bar{r})$,

$$\phi^2(r) = \phi_1^2(r) = \phi_1(\phi_1(r)) = \frac{2r-1}{1-r} < r.$$

Thus, if $r_k \in (1/2, \bar{r})$ then there is j with $\phi^j(r_k) = \phi_1^j(r_k) \leq 1/2$; i.e., $r_{k+j} \leq 1/2$ and therefore, $r_{k+j+1} \geq \phi_1(1/2) = 1$ which is a contradiction. We conclude that $\{x_n\}$ is not eventually monotonic.

(c) Because of Lemma 1, it is only necessary to show that $\mathbb{Q}^+ \subset C$. To this end, let $r_0 \in \mathbb{Q}^+$ where $r_0 = x_0/x_{-1}$. First, let us assume that both x_0 and x_{-1} are integers. Then the corresponding solution of (1) also has integer terms x_n. For each n, either $x_n \leq x_{n-1}$ or $x_n > x_{n-1}$. In the latter case, Lemma 3 implies that $x_{n+i} < x_n$ for $i \geq 1$ and in the former case, either $x_n < x_{n-1}$ or $x_n = x_{n-1}$. That is, either $r_n = 1 \in C$ or x_n must decrease. Since there are only finitely many integers involved, it follows that $r_n = 1$ or $x_n = 0$ for some n; i.e., $r_n = 1$ or 0 for a sufficiently large integer n which means that $r_0 \in C$.

Next, let x_0 and x_{-1} be any pair of positive real numbers such that $r_0 = x_0/x_{-1}$ is rational. Then $r_0 = q_0/q_{-1}$ where q_0, q_{-1} are positive integers so by the preceding argument, $r_0 \in C$ and the proof is complete.

Corollary 1. *Let $\{x_n\}$ be a solution of* (1). *Then:*

(a) *$\{x_n\}$ has period 3 eventually if and only if $x_0/x_{-1} \in \mathbb{Q}^+$ or $x_{-1} = 0$.*

(b) *$x_n = x_k(\bar{r})^{n-k}$ for some $k \geq 0$ with $x_k \leq x_0$ if and only if $x_0/x_{-1} \in D$.*

(c) *Let $x_{-1} \neq 0$. Then $x_n \to 0$ as $n \to \infty$ if and only if $x_0/x_{-1} \notin \mathbb{Q}^+$.*

(d) *$\{x_n\}$ is unstable in all cases; i.e.,* (1) *has no stable solutions.*

The next corollary is the ratios version of Corollary 1.

Corollary 2. *Let $\{r_n\}$ be a solution of* (3). *Then:*

(a) *$r_k = 0$ for some $k \geq 0$ (so r_n is undefined for $n > k$) if and only if $r_0 \in \mathbb{Q}^+$.*

(b) *For $r_0 \notin \mathbb{Q}^+$, $\{r_n\}$ is unstable.*

3 Periodic ratios and regular oscillations

Let us take a closer look at the solutions of (3) when r_0 is irrational. We begin by showing that equation (3) has periodic solutions of all possible periods except 3. With minor modifications, the next theorem applies to *eventually* periodic solutions as well.

Theorem 2. (a) *Equation* (3) *has a p-periodic solution for every $p \neq 3$.*

(b) *If* $\{r_1, \ldots, r_p\}$ *is a periodic solution of* (3) *then for the corresponding solution* $\{x_n\}$ *of* (1) *it is true that*

$$\begin{aligned} x_n &= x_0 \rho^{n/p}, \quad \text{if } n/p \text{ is an integer} \\ x_n &\leq x_0 \alpha \rho^{n/p}, \quad \text{otherwise} \end{aligned}$$

where

$$\rho = \prod_{i=1}^{p} r_i < 1, \quad \alpha = \max\{r_1, \ldots, r_p\} \rho^{-(1-1/p)} > 1.$$

Proof. (a) Let $r_1 > 1$. Then $r_2 = \phi(r_1) = \phi_2(r_1) = 1 - 1/r_1 < 1$ and

$$r_3 = \phi(r_2) = \phi_1(r_2) = \frac{1}{r_2} - 1 = \frac{1}{r_1 - 1}.$$

Though it is possible that $r_3 = r_1$, to examine potential 3-cycles, let us assume that $r_3 < 1$. Then

$$r_4 = \frac{1}{r_3} - 1 = r_1 - 2.$$

Clearly $r_4 \neq r_1$, so a period-3 solution cannot occur with two points less than 1. Since ϕ_2 maps the interval $(1, \infty)$ into $(0, 1)$, a period-3 solution cannot have two or more points greater than 1. We can also rule out a period-3 solution having all three points less than 1, since ϕ_1 is strictly decreasing on the interval (0,1). Therefore, (3) cannot have a period-3 solution. Next, we seek cycles of the form

$$r_1 > 1, \; 0 < r_k < 1, \; k = 2, 3, \ldots, p. \tag{4}$$

To explicitly determine a 2-cycle, set

$$r_1 > 1, r_2 = \phi_2(r_1) = \frac{r_1 - 1}{r_1}, \; r_3 = \phi_1(r_2) = \frac{1}{r_1 - 1} \tag{5}$$

and solve the equation $r_3 = r_1$ to obtain

$$r_1 = \frac{1 + \sqrt{5}}{2} = \gamma, \; r_2 = \frac{\sqrt{5} - 1}{\sqrt{5} + 1} = \frac{1}{\gamma^2}.$$

The number γ here is commonly referred to as the "golden mean." For explicitly listing cycles of length $p \geq 4$ that satisfy conditions (4) we need the famous Fibonacci numbers

$$y_1 = 1, \; y_2 = 2, \; y_3 = 3, \; y_4 = 5, \; y_5 = 8, \; y_6 = 13, \ldots$$

that are generated by the linear initial value problem

$$y_{n+1} = y_n + y_{n-1}, \quad y_0 = 1, \; y_{-1} = 0. \tag{6}$$

Following the pattern that was started above, namely

$$r_4 = \frac{r_1 - 2}{1 - 0}, \qquad r_5 = \frac{r_1 - 3}{2 - r_1}, \ \cdots$$

we claim that

$$r_k = \frac{y_{k-4}r_1 - y_{k-2}}{y_{k-3} - y_{k-5}r_1}, \quad k = 4, 5, \ldots, p. \tag{7}$$

with r_k given by (5) for $k = 1, 2, 3$. If we assume that (7) holds for some k, then

$$\begin{aligned} r_{k+1} &= \frac{1}{r_k} - 1 \\ &= \frac{y_{k-3} - y_{k-5}r_1 - y_{k-4}r_1 + y_{k-2}}{y_{k-4}r_1 - y_{k-2}} \\ &= \frac{y_{k-1} - y_{k-3}r_1}{y_{k-4}r_1 - y_{k-2}} \end{aligned}$$

where we used (6) for the last equality. This establishes (7) by induction. Next, using (7) we can solve the equation $r_{p+1} = r_1$ or

$$\frac{y_{p-3}r_1 - y_{p-1}}{y_{p-2} - y_{p-4}r_1} = r_1$$

to obtain the value

$$r_1 = \frac{1}{2}\left[y_{p-4} + \sqrt{y_{p-4}^2 + 4y_{p-4}y_{p-1}}\right], \quad p \geq 4$$

which together with (5), (6) and (7) completely determines the p-cycle that satisfies conditions (4) for $p \neq 3$.

(b) Without loss of generality, let $r_1 = x_1/x_0$. If $\{r_1, \ldots, r_p\}$ is a solution with period p, and

$$\rho = r_1 r_2 \cdots r_p$$

then for each positive integer k,

$$x_{kp} = r_1 r_2 \cdots r_p x_{(k-1)p} = x_{(k-1)p}\rho = \cdots = x_0 \rho^k.$$

More generally, writing $n = kp + l$ where $0 \leq l \leq p - 1$, we get

$$\begin{aligned} x_n &= r_n r_{n-1} \cdots r_{n-l+1} x_{kp} \\ &\leq \max\{r_1, \ldots, r_p\} x_0 \rho^{n/p - l/p} \\ &\leq x_0 \max\{r_1, \ldots, r_p\} \rho^{-(p-1)/p} \rho^{n/p} \end{aligned}$$

which establishes the assertion about x_n. Clearly, if $\rho < 1$ then $\alpha > 1$ since at least one of the p points of the cycle must exceed 1. Finally, $\rho < 1$ for otherwise the subsequence $\{x_0\rho^k\}$ of $\{x_n\}$

with $n = pk$ would be unbounded if $\rho > 1$, or $\{x_n\}$ would be periodic with period p if $\rho = 1$. But neither of these cases is possible.

Remark. To prove Theorem 2(a) it would have sufficed to exhibit a period-5 solution after showing that period-3 solutions are not possible. Then the proof would be complete because of the Sharkovski ordering of cycles (see Sharkovski [8], Block and Coppel [1] or Sedaghat [6]). However, using the specific nature of ϕ it was possible (and therefore, preferable) to do more and exhibit the p-cycles explicitly.

4 Chaotic ratios and irregular oscillations

It is an interesting fact that whereas the only possible period for the solutions of Equation (1) is 3, this is in fact the only period that does *not* occur for the solutions of the associated ratios equation (3)! To identify the source of this mutual exclusion, we need to look at a generalization of (1), namely, the two-parameter equation

$$x_{n+1} = |ax_n - bx_{n-1}|. \tag{8}$$

In [7] it shown that the parameter values $a = b = 1$ are bifurcation thresholds that when crossed, 3-periodic solutions occur for (1). Indeed, such solutions of (8) are shown to occur only for points (a, b) on the smooth cubic curve

$$a^3 + ab - b^3 = 1 \quad a \geq 1 \tag{9}$$

in the parameter plane that has (1,1) as an endpoint; further, (1,1) is the only point on the trace of (9) where the orbits of the 3-periodic solutions contain the origin; other parameter values on the curve (9) yield positive 3-periodic solutions for (8). We refer to [7] for additional details and a thorough study of the dynamics of (8). In the remainder of this section we show that the non-periodic solutions of (3) include chaotic solutions in the sense of Li and Yorke [3] by using the concept of snap-back repellers from Marotto [4].

Before stating the next theorem, for convenience we quote a fundamental result on chaos from [4] as Lemma 4. This result refers to the following concept: For a continuous map F of $\mathbb{R}^m$, an isolated fixed point $\bar{x}$ is a *snap-back repeller* (in the weak or non-smooth sense) if there is a sequence $\{B_k\}_{k=-\infty}^{l}$ of compact sets in $\mathbb{R}^m$ satisfying the following conditions:

(1) B_k converges to $\bar{x}$ as $k \to -\infty$;

(2) F is one-to-one on each B_k and $F(B_k) = B_{k+1}$ for every k;

(3) $\bar{x} \in \mathrm{int}(B_l)$ and $B_l \cap B_k$ is empty for $1 \leq k < l$.

Snap-back repellers are more commonly defined in the differentiable setting where a more intuitive description is possible. However, the mapping ϕ to which Theorem 3 below applies is not smooth so we need to use the more general definition of snap-back repellers that was quoted above.[1] For a proof of the following see [4] or [6].

Lemma 4. *If F has a snap-back repeller, then F is chaotic in the sense that*:

(I) *There is a positive integer N such that for each integer $p \geq N$, F has a point of period p (not necessarily stable)*;

(II) *F has a scrambled set S, i.e., an uncountable set satisfying:*

(i) *$F(S) \subset S$ and S contains no periodic points of F;*

(ii) *For every $x \in S$ every y where either $y \in S$ and $x \neq y$, or y is a periodic point of F,*

$$\limsup_{k\to\infty} \left\| F^k(x) - F^k(y) \right\| > 0,$$

(iii) *There is an uncountable set $S_0 \subset S$ such that for every $x, y \in S_0$*

$$\liminf_{k\to\infty} \left\| F^k(x) - F^k(y) \right\| > 0.$$

We note that Theorem 2(a) already establishes Part (I) above in a stronger form for our mapping ϕ. So we use Lemma 4 to prove the following:

Theorem 3. *The mapping ϕ has a scrambled set S; hence, if $\{x_n\}$ is a solution of* (1) *with initial values satisfying $x_0/x_{-1} \in S$, then the sequence $\{x_n/x_{n-1}\}$ of consecutive ratios is chaotic.*

Proof. We show that $\bar{r}$ is a snap-back repeller for ϕ. Define $I_l = [\bar{r} - \delta, \bar{r} + \delta]$ for $\delta > 0$ small enough that $I_l \subset (1/2, 1)$. Then $\bar{r} \in \mathrm{int}(I_l)$ as required by condition (3) in the definition of snap-back repeller. To complete the proof, we note that

$$\phi_1^{-1}(r) = \frac{1}{1+r} \leq 1,\ r \geq 0, \quad \phi_2^{-1}(r) = \frac{1}{1-r} \geq 1,\ 0 \leq r < 1.$$

Define $\alpha_l = \bar{r} - \delta$, $\beta_l = \bar{r} + \delta$ and $I_{l-1} = \phi_2^{-1}(I_l) = [\alpha_{l-1}, \beta_{l-1}]$ where

$$\alpha_{l-1} = \phi_2^{-1}(\alpha_l) > 1, \qquad \beta_{l-1} = \phi_2^{-1}(\beta_l) > 1.$$

Then $I_{l-1} \subset (1, \infty)$ and $I_{l-1} \cap I_l$ is empty. Further,

$$\phi_1^{-1}(I_{l-1}) = [\phi_1^{-1}(\beta_{l-1}), \phi_1^{-1}(\alpha_{l-1})].$$

Let $\beta_{l-2} = \phi_1^{-1}(\alpha_{l-1})$, $\alpha_{l-2} = \phi_1^{-1}(\beta_{l-1})$ and define $I_{l-2} = [\alpha_{l-2}, \beta_{l-2}]$. Then

$$\alpha_{l-1} > 1 > \bar{r} \Rightarrow \beta_{l-2} = \phi_1^{-1}(\alpha_{l-1}) < \phi_1^{-1}(\bar{r}) = \bar{r}$$

[1] Refer to the paper "The Li-Yorke theorem and infinite discontinuities, J. Math. Analysis and Appl. 296 (2004) 538-540 (or its pre-publication copy at http://www.discretedynamics.net/Articles/articles.htm) for a correction on the use of snap-back repellers.

so that

$$I_{l-2} \subset (0, \bar{r}). \tag{10}$$

Next, we define

$$I_{l-3} = \phi_1^{-1}(I_{l-2}) = [\phi_1^{-1}(\beta_{l-2}), \phi_1^{-1}(\alpha_{l-2})] = [\alpha_{l-3}, \beta_{l-3}]$$

and notice that $\alpha_{l-3} > \phi_1^{-1}(\bar{r}) = \bar{r}$ and $\beta_{l-3} \leq 1$. Hence,

$$I_{l-3} \subset (\bar{r}, 1]. \tag{11}$$

Now, if for $j \geq 2$ we define the following sequence

$$I_{l-j} = \phi_1^{-1}(I_{l-j+1}) = [\alpha_{l-j}, \beta_{l-j}]$$

where

$$\alpha_{l-j} = \phi_1^{-1}(\beta_{l-j+1}) = \frac{1}{1+\beta_{l-j+1}},$$
$$\beta_{l-j} = \phi_1^{-1}(\alpha_{l-j+1}) = \frac{1}{1+\alpha_{l-j+1}}$$

then from (10) and (11) it follows that $0 < \alpha_{l-j}, \beta_{l-j} \leq 1$ for $j \geq 2$ and thus, the intervals I_{l-j} are well-defined. In fact, if $\phi_1^{-2}(r) = \phi_1^{-1}(\phi_1^{-1}(r))$, then

$$\alpha_{l-2j} = \phi_1^{-2}(\alpha_{l-2j+2}) > 0, \quad \beta_{l-2j} = \phi_1^{-2}(\beta_{l-2j+2}) < \bar{r}.$$

We claim that

$$\alpha_{l-2j}, \beta_{l-2j} \to \bar{r} \quad \text{as} \quad j \to \infty. \tag{12}$$

If this is true, then

$$\alpha_{l-2j-1} = \phi_1^{-1}(\beta_{l-2j}) \to \bar{r}, \quad \beta_{l-2j-1} = \phi_1^{-1}(\alpha_{l-2j}) \to \bar{r}$$

and it follows that the compact intervals I_{l-j} converge to $\bar{r}$. From this and the fact that ϕ_1 is strictly decreasing on $(0, 1]$ it necessarily follows that $\bar{r}$ is a snap-back repeller (in the definition of snap-back repeller we may take $k \geq 2$ to be the least integer j for which $I_{l-j} \cap I_l$ is non-empty).

To prove the claim (12), it suffices to show that if $s \in (0, \bar{r})$ then

$$\lim_{n\to\infty} \phi_1^{-2n}(s) = \bar{r}. \tag{13}$$

To see this, observe that if $r < \bar{r}$ then

$$\phi_1^{-2}(r) = \frac{1+r}{2+r} > r\frac{1+1/\bar{r}}{2+\bar{r}} = r.$$

That is,

$$\phi_1^{-2}(r) > r \quad \text{for} \quad r \in (0, \bar{r}). \tag{14}$$

Further, ϕ_1^{-2} is an increasing function since $d\phi_1^{-2}/dr = 1/(2+r)^2 > 0$. It follows that

$$r < \phi_1^{-2}(r) < \bar{r} \quad \text{for} \quad r \in (0, \bar{r})$$

since $\phi_1^{-2}(\bar{r}) = \bar{r}$, and together with (14), this proves that $\{\phi_1^{-2n}(s)\}$ is an increasing sequence in $(0, \bar{r})$. Therefore, (13) is true, which proves (12) and thus completes the proof that $\bar{r}$ is a snap-back repeller. The proof of the theorem is completed upon applying Lemma 4.

Remarks. 1. (Elements of D) Each interval I_l in the proof of Theorem 3 contains an inverse image of $\bar{r}$, i.e., an element of the set D mentioned earlier from which all eventually monotonic solutions arise. It is possible to explicitly list these particular elements of D. Starting with $\bar{r}$, we compute

$$r^* = \phi_2^{-1}(\bar{r}) = \frac{1}{1-\bar{r}} = \frac{2}{3-\sqrt{5}}$$

Next, we obtain successive inverse images $\phi_1^{-n}(r^*)$ for all positive integers n. It can be shown by straightforward induction that

$$\phi_1^{-n}(r^*) = \frac{2y_n + y_{n-2} + y_{n-2}\sqrt{5}}{2y_{n+1} + y_{n-1} + y_{n-1}\sqrt{5}}$$

where y_n is the n-th Fibonacci number as generated by the difference equation (6).

2. The mapping ϕ is a one-dimensional semiconjugate factor of the mapping

$$F(x, y) = (|x - y|, x)$$

namely, the standard vectorization or the unfolding of Eq.(1). The ratios may be naturally considered a link between ϕ and F. We have seen the usefulness of this semiconjugate relationship above in describing the asymptotic behavior of Eq.(1). For more on one-dimensional semiconjugates in general as well as other examples, see [6].

References

[1] L. Block and W.A. Coppel, "Dynamics in One Dimension," Springer, New York, 1992

[2] V.L. Kocic and G. Ladas, "Global Behavior of Nonlinear Difference Equations of Higher Order with Applications," Kluwer Academic, Dordrecht, 1993

[3] Li, T.Y. and J.A. Yorke, Period three implies Chaos, *Amer. Math. Monthly*, 82 (1975), 985-992

[4] Marotto, F.R., Snap-back repellers imply chaos in $\mathbb{R}^n$, *J. Math. Analysis and Appl.* 63 (1978), 199-223

[5] Sedaghat, H., The global stability of equilibrium in a nonlinear second order difference equation, *Int. J. Pure and Appl. Math.* (2003, to appear)

[6] Sedaghat, H., "Nonlinear Difference Equations: Theory with Applications to Social Science Models," Kluwer Academic, Dordrecht, 2003
[7] Sedaghat, H., On the 2-parameter absolute difference equation (to appear)
[8] Sharkovski, A.N., Coexistence of cycles of a continuous mapping of the line into itself, *Ukrain. Math. Zh.* 16 (1964), [in Russian]