MATH 195: Gödel, Escher, and Bach (Spring 2000)

Problem Set 2: *The MU-Puzzle*

To be discussed Thu, January 25 (1-4) and Tuesday, January 30 (rest)

These problems range from the relatively straightforward to rather difficult. Don't get frustrated if you're stumped by one or another, but go as far as you can on them by Tuesday.

2.1. Starting with **MI,** derive **MIU** in two different ways**.**

2.2. Starting with **MI**, derive **MIUUI**.

2.3. Using the **MU-Puzzle** program (or just pen and paper), generate 10 different theorems.

2.4. Using the **MU-Puzzle** program (see instructions) or just a pen and paper, generate 20 different theorems systematically.

- **a.** Theorem #1 is **MI** (it's also an axiom)
- **b.** Apply RULES I through IV[1] to Theorem #1 (some may not be applicable), generating new theorems. Number them consecutively.
- **c.** Apply RULES I through IV to the theorems you generated in **Step b**. Give them numbers as well.
- **d.** Apply RULES I through IV to the theorems you generated in **Step c**. . . and so forth, until you've generated at least 20 theorems.
- **e.** What is the 20^{th} theorem?
- **f.** You've been acting like a machine. Now break out of the system and examine the theorems you've generated. What generalities, if any, do you notice?
- **g.** If you managed to generate **MU**. . . congratulations! If not, then how far would you need to go to convince yourself that the *MIU-system* is incapable of doing so?
- **h.** Circle all theorems that consist solely of **I**'s plus one **M**. How many **I**'s are in these theorems? Give a metarule that describes how you could recognize such theorems at a glance.

2.5. Prove that if **MI** is the sole axiom, the *MIU-system* can produce no theorem that does not begin with **M**.

2.6. Prove that if **MI** is the sole axiom, the *MIU-system* cannot produce **MU** in less than five steps.

2.7. Either generate **MU** within the *MIU-system* using **MI** as the sole axiom or prove that you cannot.

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[1] You may be able to apply RULE III more than once to a previous theorem. In such cases, proceed left to right. For example, applying the rule to **MIIIIII** would generate, in order, **MUIII**, **MIUII**, **MIIUI**, and **MIIIU**.

2.8. Consider the *MIU-system* with **MII** as the sole axiom. Is **MI** a theorem?

2.9. Consider the *MIU-system* with **MU** as the sole axiom. Is **MI** a theorem?

2.10. Can you think of a string that can serve as the sole axiom in the *MIU-system* and permit you to derive **MU**?

2.11. Consider the *MIU-system* with **MIUIIII** as the sole axiom. Is **MI** a theorem? Is **MU**?

2.12. Can you generate any generality about what sole axioms permit the derivation of **MU**? Of **MI**?

2.13. Change a rule of the *MIU-system* so that **MU** could be derived from **MI**.

2.14. Consider the string **MIUIIUIIIUIIIIU** (each set of **I**'s is one more than the last). Describe how to derive this string within the *MIU-system* using **MI** as the sole axiom? If you need a hint, just ask. (Unasked for hint: Don't try solving this without insight, i.e. by