Graphs with $\chi = \Delta$ have big cliques

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Joint with Landon Rabern
Slides available on my webpage

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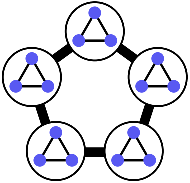

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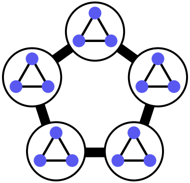

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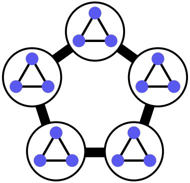

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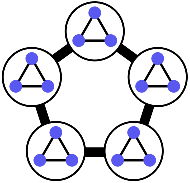

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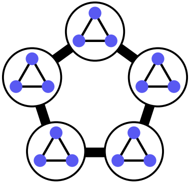

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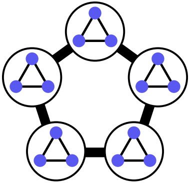

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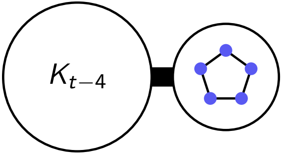

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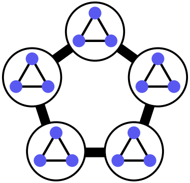

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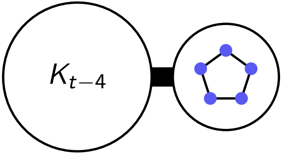

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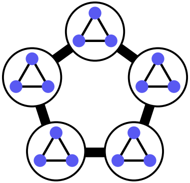

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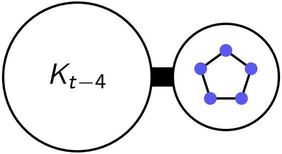

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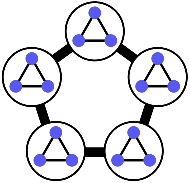

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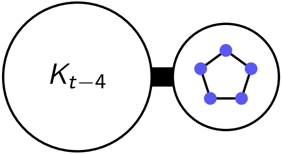

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- If $\Delta(G - I) = \Delta(G) - 1$, then $G - I$ is a smaller counterexample, contradiction!

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Clubs and Clubhouses

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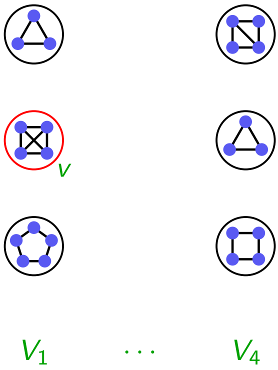

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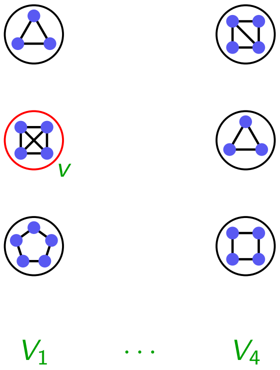

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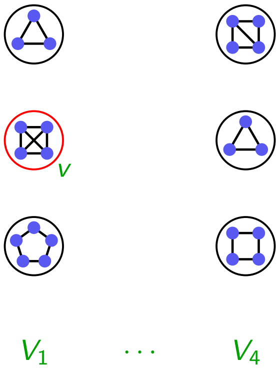

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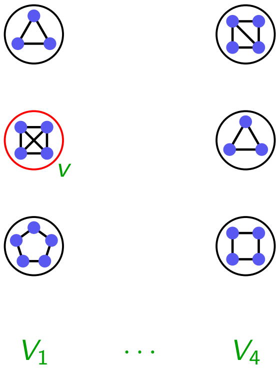

Lem: Every Δ-critical graph with $\Delta = 13$ has a Mozhan partition.

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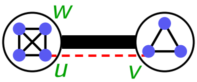

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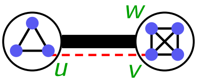

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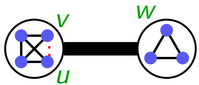

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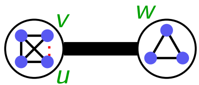

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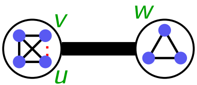

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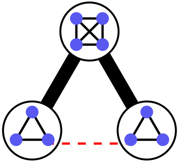

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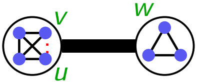

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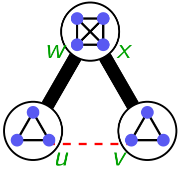

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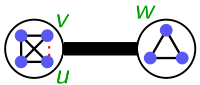

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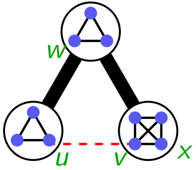

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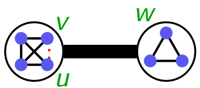

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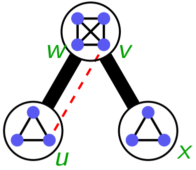

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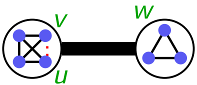

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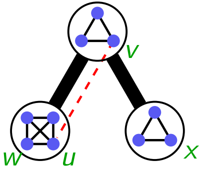

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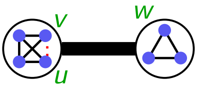

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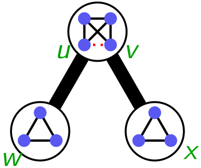

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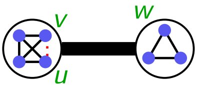

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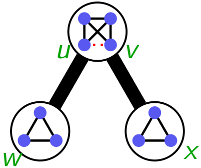

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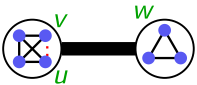

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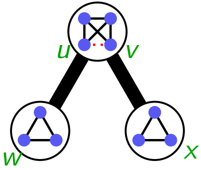

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